Lecture Notes on Atmospheric Thermodynamics and Cloud Physics

Harshit Prashant Dhanwalkar (SC21B164)[1]*

Abstract
Notes of Lectures plus addional information from books.

[1] MTech, Earth System Sciences (ESS), 1st year, Department of Physics, Indian Institute Of Spacescience and Technology (IIST)
***email**: harshitpd1729@gamil.com

Contents

1. Lecture 1 09/08/2024

- System: Specific chunk of matter we are interested in.
- Surrounding: Everything else in the universe outside of the system.
- Air parcel: System with the following assumptions:
    - Properties within the air parcel are uniform.
    - Boundaries are closed, so that matter/mass is not exchanged with the surroundings.

Question 1.1: How do system interact with surrounding?
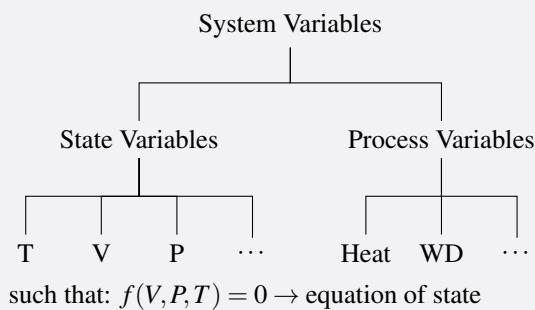
- Matter
- Radiation

Question 1.2: Classify system on the basis of how matter interact with surround?

$\Rightarrow$ We can classify system into 3 types on the basis of how matter in system interact with it's surrounding:
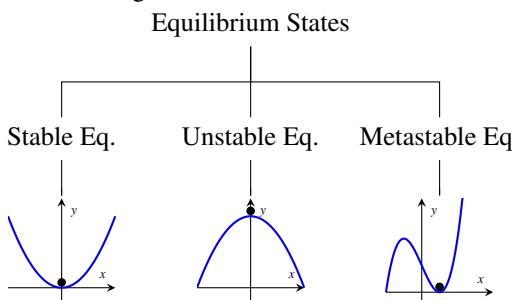
- Open
- Closed
- Isolated

Question 1.3: How do we characterize a system?

System Variables

State Variables   Process Variables

T   V   P $\cdots$ Heat   WD $\cdots$

such that: $f(V,P,T) = 0 \rightarrow$ equation of state

1.1 Equilibrium State
Equilibrium: No change in the system if the surrounding doesn't change.

Equilibrium States

Stable Eq.   Unstable Eq.   Metastable Eq.

1.2 Transformation

Transformation Process

Reversible   Irriversible
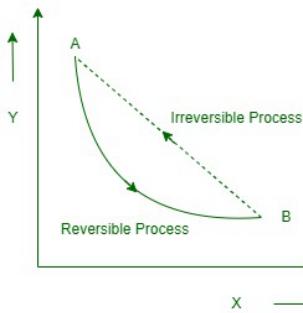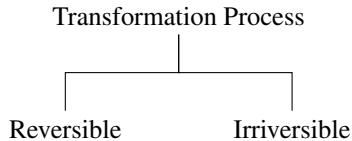
Figure 1. Reversible and Irreversible Process

1.3 Exact and Inexact differentiable
Let z be a function of x and y, exact differentuable equation:

$$dz = \left(\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial z}{\partial y}\right)dy \tag{1}$$

$$dz = \int_i^f \delta z = z(x_f, y_f) - z(x_i, y_i) \tag{2}$$

$\therefore dT, dP, dV$ are exact diffferenrtiable.
$\delta Q, \delta W$ are inexact diffferenrtiable.
line intregral:

$$\oint \delta z = 0$$

$\rightarrow z$ is stable variable and it is exact differentiable iff it's reversible

$$\therefore \oint dT = 0, \oint dP = 0, \oint dq \neq 0, \oint dw \neq 0$$

1.4 Exntensive and Intrincive variables
- **Entrisive**: Depends on size of system. E.g. Volume.
- **Intrincive**: Independent of size of system. E.g. Temperature.

Any variable divided by mass gives intrincive.
For e.g. specific volume $(\alpha) = \frac{volume}{mass}$.
Similiarly,
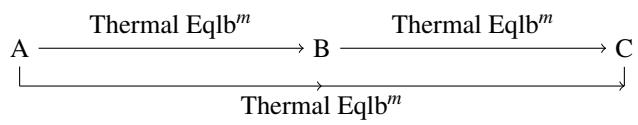
$$p = \frac{P}{m}, q = \frac{Q}{m}, w = \frac{W}{m}$$

1.5 Laws of Thermodynamics
1. 0th law of thermodynamics:

    Temperature $\rightarrow$ Quantity that determines the direction of heat flow. If two objects are in thermal contact and there is no net heat transfer, then the system is said to be in **thermal equilibrium**.

$$A \xrightarrow{\text{Thermal Eqlb}^m} B \xrightarrow{\text{Thermal Eqlb}^m} C$$

$$A \xrightarrow{\hspace{9cm}} C$$
$$\text{Thermal Eqlb}^m$$

2. Lecture 2 14/08/2024

Bulk properties $\rightarrow$ microscopic properties $\rightarrow$ which can be linked to microscopic properties

For e.g. P is exerted due to random motion and colllion of moleules with each other and on the walls of container.
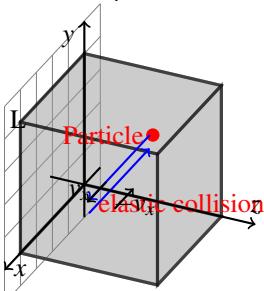
2.1 Kinetic molecular therory of gases
Conditions:

- Molecules are in **random motion**.
- Collions between molecules with each other and wall of container are **elastic** in nature ensuring **no K.E. loss**.

Derivation of av. K.E.

Particles travels distance equal to lenght of box (L) with velocity V_x, V_y and V_z.

Force excerted by the molecules on the face:

$$F = \frac{\Delta P}{\Delta t} = \frac{\Delta mv}{\Delta t} = m\frac{\Delta v}{\Delta t} \tag{3}$$

Change in velocity

$$\Delta v_x = v_x - (-v_x) = 2v_x \tag{4}$$
$$\Delta v_y = 0 \tag{5}$$
$$\Delta v_z = 0 \tag{6}$$
$$\implies \Delta V = 2v_x \tag{7}$$

$$\therefore F = \frac{m(2v_x)}{\Delta t} \tag{8}$$

We also have,

$$\Delta t = \frac{2L}{v_x} \tag{9}$$

from Eq. (8) and Eq. (9), we get:

$$F = \frac{m(2v_x^2)}{2L} = \frac{mv_x^2}{L} \tag{10}$$

For N number of molecules and average velocity of all molecules moving in x-direction $\bar{v}_x$

$$\therefore F = \frac{Nm(\bar{v}_x^2)}{L} \tag{11}$$

Pressure (P) exerted on walls of container:

$$P = \frac{F}{A} = \frac{Nm(\bar{v}_x^2)}{L \times L^2} = \frac{Nm(\bar{v}_x^2)}{V} \tag{12}$$
$$\implies PV = Nm(\bar{v}_x^2) \tag{13}$$

When we consider the velocity of molecules in all directions (v_{tot}).

$$v_{tot}^2 = \bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2 \tag{14}$$
$$\implies \bar{v}_{tot}^2 = 3\bar{v}_x^2 \tag{15}$$

$\therefore$ The Pressue(P) becomes:

$$PV = \frac{1}{3}Nm\bar{v}_{tot}^2 \tag{16}$$
$$3PV = Nm\bar{v}_{tot}^2 \tag{17}$$
$$\frac{3}{2}PV = \frac{1}{2}Nm\bar{v}_{tot}^2 \tag{18}$$
$$\frac{3}{2}PV = N \times \left(\frac{1}{2}m\bar{v}_{tot}^2\right) \tag{19}$$
$$\implies \frac{3}{2}PV = N \times (K.E.)_{av} \tag{20}$$
$$\tag{21}$$

Using Ideal gas equation : $PV = Nk_BT$, where k_B is boltzman constant

$$\frac{3}{2}(Nk_BT) = N \times (K.E.)_{av} \tag{22}$$
$$\implies (K.E.)_{av} = \frac{3}{2}k_BT \tag{23}$$

2.2 Ideal gas
1. Molcules are in random motion.
2. During the motion of **molecules do not exert force, except when they collide with each other or the walls of container**. This can also be stated as there is **no force of attraction between mocecules**.
3. The collisions between molecules are **elastic**.
4. **Sum of the volume of molecules is negligible** comapered to volume of container.
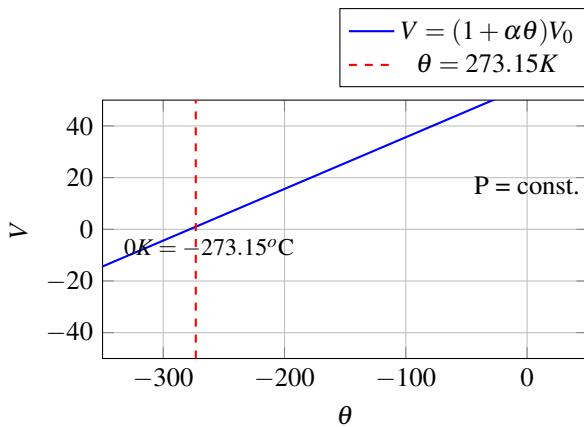
2.3 Early experiments and laws
1st **Law of Gay-Lussac**

Increase of volume of an ideal gas at **constant pressure** is proportional to incrase in temperature and also to the volume occupied by the gas at 0^oC

$$dV \propto V_0 d\theta \tag{24}$$
$$dV = \alpha V_0 d\theta \tag{25}$$

Figure 2. Plot of V vs. θ

where α is volume coeffient of thermal expansion.

$$\alpha = \frac{1}{d\theta}\frac{dV}{V_0} = \frac{1}{273} \tag{26}$$

$$\int_{V_0}^{V} dV = \int_{0^oC}^{\theta} \alpha V_0 d\theta \tag{27}$$

$$V - V_0 = \alpha V_0(\theta - 0^oC) \tag{28}$$

$$\Rightarrow V = (1 + \alpha\theta)V_0 \tag{29}$$

2st **Law of Gay-Lussac**

An Ideal gas kept at constant volume, then the increase in pressure is proportional to increase in termperature and pressure at 0^oC.

$$dP \propto P_0 d\theta \tag{30}$$

$$dP = \beta P_0 d\theta \tag{31}$$

where β is pressure coeffient of thermal expansion.

$$\beta = \frac{1}{d\theta}\frac{dP}{P_0} = \frac{1}{273} \tag{32}$$

$$\int_{P_0}^{P} dP = \int_{0^oC}^{\theta} \beta P_0 d\theta \tag{33}$$

$$P - V_0 = \beta P_0(\theta - 0^oC) \tag{34}$$

$$\Rightarrow P = (1 + \beta\theta)P_0 \tag{35}$$

3. Lecture 3 16/08/2024

Question 3.1: Why do we need Kinetic theory of gases?

$\Rightarrow$ Kinetic therory of gases connects microscopic properties to macroscopic properties of gases.

3.1 Another form of Gay-Lussac's Law
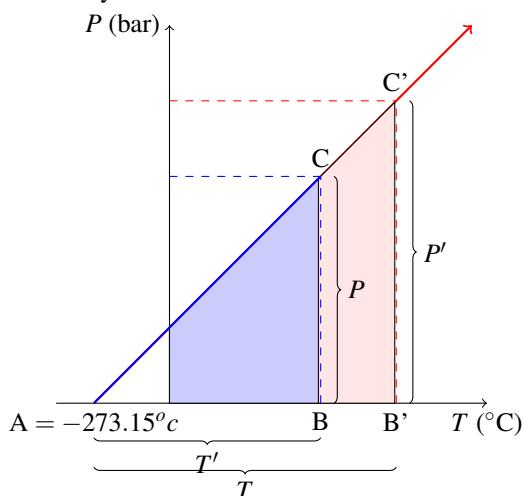
From similiarity of triangle;

$$\frac{BC}{AB} = \frac{B'C'}{AB'}$$
$$\frac{V}{T} = \frac{V'}{T'}$$
$$\frac{V}{V'} = \frac{T}{T'}$$

for P is Constant
Similiarlly for Pressure at constant Volume

From similiarity of triangle;

$$\frac{BC}{AB} = \frac{B'C'}{AB'}$$
$$\frac{P}{T} = \frac{P'}{T'}$$
$$\frac{P}{P'} = \frac{T}{T'}$$

for V is Constant

3.2 Boyle's Law
Boyle's Law states that, at constant temperature, the pressure of a given amount of gas is inversely proportional to its volume. Mathematically, it is expressed as:

$$P \propto \frac{1}{V} \tag{36}$$
$$PV = P'V' \tag{37}$$
$$PV = const. \tag{38}$$

where P is the pressure of the gas, and V is its volume. This implies that if the volume of a gas increases, its pressure decreases, and vice versa, as long as the temperature and the amount of gas remain constant.

3.3 Avagadro's Law
Avogadro's Law states that, at the same temperature and pressure, equal volumes of all gases contain the same number of molecules. Mathematically, it is expressed as:

$$V \propto n \tag{39}$$

where V is the volume of the gas, and n is the number of moles of the gas. This implies that the volume of a gas is directly proportional to the number of moles, provided temperature and pressure are constant.
For one mole of gas contains $6.023 \times 10^{23} mol^{-1}$ of particles.

3.4 Ideal gas Law
System defined by (P, V, T) undergo change shown in following Fig.3
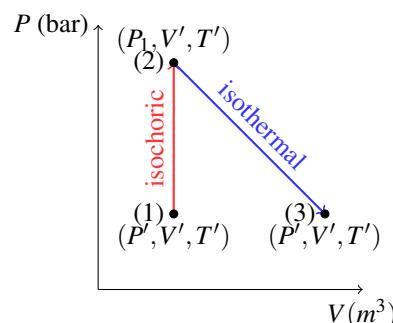
Figure 3. Diagram illustrating different thermodynamic processes in a P-V diagram.

Apply Gay-Lussac law $(1) \rightarrow (2)$:

$$P_1 = P\frac{T'}{T} \tag{40}$$

Apply Boyle's law $(2) \rightarrow (3)$:

$$P'V' = P_1V$$

from eq.40 we get

$$P'V' = \left(\frac{PT'}{T}\right)V$$

after rearranging we get:

$$\frac{P'V'}{T'} = \frac{PV}{T}$$

$$\frac{PV}{T} = \text{const}$$

$$PV = AT$$

where $A = nR^*$

R^* is universal gas constant $(8.314 J K^{-1} mol^{-1})$

$$\implies PV = nR^*T \tag{41}$$

3.5 Van Der Waal's equation

$$\left[P + a\left(\frac{n}{V}\right)^2\right]\left[V - (nb)\right] = nR^*T \tag{42}$$

Where
a = coeff. which characterises the intermolecular forces = $1.35 \times 10^5 J m^3 K^{-1} mol^{-2}$
b = coeff. which accounts for effective volume occupied by molecules = $3.64 \times 10^{-2} m^3 K^{-1} mol^{-1}$
When $nb \rightarrow 0$ or $\frac{n}{V} \rightarrow 0$ Van Der Waal equation $\rightarrow$ Ideal gas equation.

3.6 Meteorological form of Ideal gas
Let there be n-kilomols particles/molecules of gas.
Therefore the combination of ith component of the gas can be given by:

$$n = \sum_{i=1}^{k} n_i \tag{43}$$

The total mass of sample in kg:

$$M = \sum_{i=1}^{k} n_i m_i \tag{44}$$

where m_i represents molar mass of ith particle/molecule in a sample.
Using Ideal gas equation:

$$PV = nR^*T \tag{45}$$

$$\frac{PV}{M} = \frac{nR^*T}{M} \tag{46}$$

$$P\alpha = \frac{n}{M}R^*T \tag{47}$$

since $\frac{V}{M} = \alpha$, and called specfic volume.

$$P\alpha = \frac{\sum_{i=1}^{k} n_i}{\sum_{i=1}^{k} n_i m_i}R^*T \tag{48}$$

$$P\alpha = \frac{R^*T}{\bar{m}} \tag{49}$$

$$\Rightarrow P\alpha = R_dT \tag{50}$$

where $\bar{m}$ is mean molar mass and given by $\frac{\sum_{i=1}^{k} n_i m_i}{\sum_{i=1}^{k} n_i}$ and $R_d = \frac{R^*}{\bar{m}}$ and unit $JKg^{-1}K^{-1}$

Since we know $\alpha = \frac{1}{\rho}$, where ρ is density of gas.

$$P\alpha = R_dT \tag{51}$$

$$\Rightarrow P = \rho R_dT \tag{52}$$

Question 3.2: what is R_d?

$\Rightarrow R_d \rightarrow$ is a specific gas constant where d stands for dry air and this constant is not universal, varies with time and conditons over a particular place.

$$R_d = \frac{R^*}{\bar{m}}$$

and unit $JKg^{-1}K^{-1}$

3.7 Composition of Earth's Atmosphere

Gas	Fraction/Volume	Molecular Mass
N_2	78.1%	28.01
O_2	20.9%	31.999
Ar	0.93%	39.9

Table 1. Composition of Earth's Atmosphere.

We haven't take water vaours($H_2O(v)$), Carbon diaoxide(CO_2) and Ozone(O_3) becasue these gaes are highly variable w.r.t time and geography.

Question 3.3: Find R_d for Earth Atmosphere?

$$\Rightarrow R_d = \frac{R^*}{\bar{m}}$$

$$= \frac{8.314 \times 10^3}{28.96}$$

$$= 0.287085$$

$$= 287.085 Jkg^{-1}K^{-1}$$

---

4. Lecture 4 19/08/2024

$CO_2 \rightarrow$ radiation trapping process $\rightarrow$ absorb radiation in IR region of spectrum.

Question 4.1:

1. **Determine the gas constant for the atmosphere of Venus which consists of 95% CO_2 and 5% N_2 by volume.**

2. **The mean surface temperature of Venus is $740K$ as compared to $288K$ surface temperature of Earth. The surface pressure on Venus is 90 times that on Eath. By what factor the density of near surafce Venusin atmosphere is greter than the Earth?**

$\Rightarrow$ Solution:

1. Mean molar mass of gases of atmosphere of Venus:

$$\bar{m} = \frac{\sum_{i=1}^{k} n_i m_i}{\sum_{i=1}^{k} n_i}$$

$$\bar{m} = 0.95 \times 46 + 0.05 \times 28$$

$$\bar{m} = 43.2$$

Calculate Specific Molar const.(R_V) for Venus:

$$R_V = \frac{R^*}{\bar{m}}$$

$$\Rightarrow R_V \approx 192 \; JK^{-1}Kg^{-1}$$

2. Surface Temperature of Venus = $T_V = 740K$
   Surface Temperature of Earth = $T_V = 740K$
   Surface Pressure of Earth = P_E
   Surface Pressure of Venus = $P_V = 90 \times P_E$

$$P_V = \rho_V R_V T_V$$

$$P_E = \rho_V R_E T_E$$

$$\frac{P_V}{P_E} = \frac{\rho_V R_V T_V}{\rho_V R_E T_E}$$

$$90 = \frac{\rho_V}{\rho_E} * \frac{192}{287} * \frac{740}{288}$$

$$\frac{\rho_V}{\rho_E} = 52.36$$

Venus's atmosphere is 52.36 times densier than Earth's atmoshpere.
$\Rightarrow \rho_E = 1.23 \; kgm^{-3}$ and $\rho_V = 65.97 \; kgm^{-3}$

Question 4.2: Why CO_2 have radiation trapping affinity but not gases like N_2?

$\Rightarrow$ CO_2 has radiation trapping affinity because its molecular structure allows it to absorb and re-emit IR radiation, contributing to the greenhouse effect. This is due to its vibrational modes that change the molecule's dipole moment. In contrast, N_2, with its symmetric diatomic structure, cannot absorb infrared radiation effectively, as its vibrations do not change the dipole moment, making it non-contributory to the greenhouse effect.

4.1 Pressure

Units of Pressure

Pressure are usually expressed in thefollwing units:

1 bar = 1.013×10^5 Pa
10^5 Pa = 1000hPa = 1000 mbar

Question 4.3: Why does ozone layer depletion happen primarily over the South Pole in Antarctica?

$\Rightarrow$ Ozone layer depletion happen primarily over the South Pole becasue of following reasons:

- **Polar Stratospheric Clouds (PSCs):** During the Antarctic winter, temperatures drop below $-75°C$, leading to the formation of PSCs. These clouds facilitate chemical reactions that convert inactive chlorine compounds into reactive forms, which destroy ozone.

- **Isolation of the Polar Vortex:** The strong polar vortex over Antarctica isolates air, keeping temperatures low and trapping ozone-depleting chemicals within the vortex.

- **Sunlight and Ozone Destruction:** In Antarctic spring, returning sunlight provides energy for reactions between chlorine radicals and ozone, leading to significant ozone depletion and the formation of the "ozone hole."

- **Comparison with the Arctic:** The Arctic has a weaker and less stable polar vortex, resulting in less dramatic ozone depletion compared to Antarctica.

4.2 Mass of the Atmosphere

At any point in the atmosphere, atmosphere above will exert a downward force due to gravitational force

$$F = \rho g \tag{53}$$

$$F = \int_0^\infty \rho g \, dz \tag{54}$$

We know $P = $ Force/(unit area), assuming gravity g_0 remain constant, we get:

$$\Rightarrow P_s = g_0 \int_0^\infty p\,dz \tag{55}$$

where P_s is vertically integrated and have unit kgm^{-2}

Question 4.4: Globally average surface pressure of earth is 985hpa. Estimate mass of atmossphere.

$\Rightarrow$ Given $985hpa = 985 \times 10^2\,Pa$
surface area(sa) $= 4\pi r^2$
where r is radius of earth $= 6400km$
we know, pressure$(p) = \frac{force}{sa}$

$$p = \frac{mg}{4\pi r^2}$$
$$986 \times 10^2 = \frac{m \times 9.81}{4 \times \pi \times (6400 \times 10^3)^2}$$
$$m = 5.168172908 \times 10^{18} kg$$

$\therefore$ The approximate mass of atmosphere is equal to 5.1708×10^{15}

Question 4.5: The average atmospheric pressure on surface of Mars is $6hPa$ and raduis $3400km$. Find mass of Mars.

$\Rightarrow$ Given Surface pressure of Mars $P = 6hPa = 6 \times 10^2\,pa$
Radius of Mars$R = 3400km = 3.4 \times 10^6 m$
surface area(sa) $= 4\pi r^2$
where r is radius of earth.
we know, pressure$(p) = \frac{force}{sa}$

$$p = \frac{mg}{4\pi r^2}$$
$$986 \times 10^2 = \frac{m \times 9.81}{4 \times \pi \times (6400 \times 10^3)^2}$$
$$m = 5.168172908 \times 10^{18} kg$$
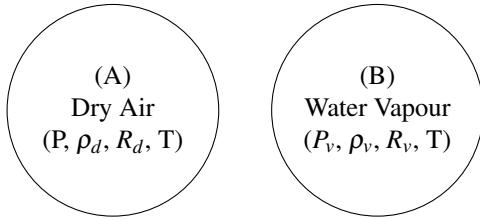
$\therefore$ The approximate mass of atmosphere is equal to $5.168172908 \times 10^{18} kg$

5. Lecture 5 21/08/2024

In previous lecture, We got equation for Ideal gas Eq.(52), which only deals with dry air.
We didn't incooperated moisture!

(A)
Dry Air
(P, ρ_d, R_d, T)

(B)
Water Vapour
(P_v, ρ_v, R_v, T)

where P_v is vapour pressure and given by

$$P_v = \rho_v R_v T \tag{56}$$

$$R_v = \frac{R^*}{\bar{m}} = \frac{8.314 \times 10^3}{18.01} = 461.63 J K^{-1} kg^{-1} \tag{57}$$

Note: Water vapour is not same as moisture, misture is mixture of air and water vapour

5.1 Dalton's law of partial pressure

Dalton's law of partial pressure states that total pressure exerted by the mixture of gas is equal to the sum of partial pressure exerted by individual contituent at a given temperature.

$$P = P_d + e \tag{58}$$

Where

$$P = \text{ Total pressure excerted by all gases in mixture}$$
$$P_d = \text{Pressure exerted by dry air}$$
$$e = P_v = \text{Vapour pressure}$$

Substituting Eq.(52) and Eq.(56) in Eq.(58), we get:

$$P = \rho_d R_d T + \rho_v R_v T \tag{59}$$
$$P = (\rho_d R_d + \rho_v R_v)T \tag{60}$$

5.2 Humidity

We define humidity using following parameters:

1. Mixing ratio:

$$\omega = \frac{\text{Mass of water vapour}}{\text{Mass of dry air}} = \frac{M_v}{M_d} \tag{61}$$
$$= \frac{\text{Density of water vapour}}{\text{Density of dry air}} = \frac{\rho_v}{\rho_d} \tag{62}$$

Unit of mixing ratio is g/Kg

2. Specific humidity:

$$\omega = \frac{\text{Mass of water vapour}}{\text{Mass of dry air} + \text{Mass of water vapour}} \tag{63}$$

$$= \frac{M_v}{M_d} \tag{64}$$

$$= \frac{\text{Density of water vapour}}{\text{Density of dry air} + \text{Density of water vapour}} \tag{65}$$

$$= \frac{\rho_v}{\rho_d + \rho_v} \tag{66}$$

$$= \frac{\rho_v}{\rho} \tag{67}$$

$w \approx q, \because$ mass of water vapour $\ll$ masss of dry air

$$\omega = \frac{\rho_v}{\rho_d} \tag{68}$$

$$= \frac{e/R_v T}{P_d/R_d T} \tag{69}$$

$$= \frac{e\varepsilon}{P_d} \tag{70}$$

$$= \frac{e\varepsilon}{P - e} \tag{71}$$

$$\approx \frac{e\varepsilon}{P} \tag{72}$$

where $\varepsilon = \frac{R_d}{R_v} = 0.621$
Similiary,

$$q = \frac{\rho_v}{\rho_v + \rho_d}$$
$$= \frac{e\varepsilon}{P - (1-\varepsilon)e}$$
$$\approx \frac{e\varepsilon}{P}$$
$$q \approx w$$

5.3 Ideal gas equation for moist gas

Total pressure

$$P = P_d + e$$
$$= \rho_d R_d T + \rho_v R_v T$$
$$= \rho_d R_d T \left[1 + \frac{\rho_v R_v}{\rho_d R_d}\right]$$
$$= \rho_d R_d T \left[1 + \frac{\rho_v}{\rho_d} \cdot \frac{R_v}{R_d}\right]$$
$$= \rho_d R_d T \left[1 + \frac{\rho_v}{\rho_d} \cdot \frac{1}{\varepsilon}\right]$$
$$= \rho_d R_d T \left[1 - \left(1 - \frac{1}{\varepsilon}\right)\frac{\rho_v}{\rho}\right]$$
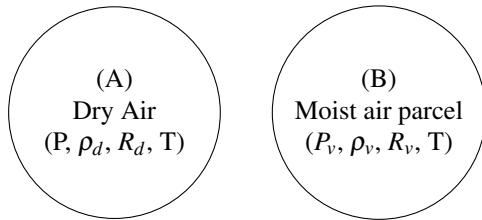
$$\therefore P = \rho_d R_d T \left[1 - \left(1 - \frac{1}{\varepsilon}\right) \cdot q\right] \tag{73}$$

Virtual Temerature

$$\Rightarrow T_v = \frac{P}{\rho_d R_d} = T\left[1 - \left(1 - \frac{1}{\varepsilon}\right) \cdot q\right] \tag{74}$$

Therefore from Eq.(73) and Eq.(74), we get:

$$P = \rho_d R_d T_v \tag{75}$$

(A)
Dry Air
(P, ρ_d, R_d, T)

(B)
Moist air parcel
(P_v, ρ_v, R_v, T)

$\rho_d > \rho_m \xrightarrow{\triangle} \rho_d = \rho_m$, where $\triangle$ represents heat.

Question 5.1: On a summer day the AC breaks down and the air in the classroom becomes warm and muggy with a vapour pressure of 20 hPa and a temperature of 25°C.
a.   If the volume of the classroom is 40m^3. How much water is present in the room in vapour form?
b.   If pressure of the room is $900hPa$ then what is virtual temerpature of the air?

$\Rightarrow$ a. We have,

$$P_v = \rho_v R_v T$$
$$20 \times 10^2 = \rho_v \times 461.62 \times 298$$
$$\rho_v = 0.0145\,kg/m^3$$
$$\frac{m}{V} = 0.0145$$
$$\frac{m}{40} = 0.0145$$
$$m \approx 0.58149\,kg$$

$\therefore$ Amount of water vapour in room is $0.58148976\,kg$.

$\Rightarrow$ b. Given, $P = 900\,\text{hPa} = 90000\,\text{Pa}$
Let T_v be the virtual temperature. We know:

$$T_v = T\left(1 + 0.61\frac{P_v}{P}\right)$$

Substituting the values:

$$T_v = 298\left(1 + 0.61 \times \frac{2000}{90000}\right)$$
$$T_v = 298 \times 1.01356$$
$$T_v \approx 302.04\,\text{K}$$

$\therefore$ The virtual temperature of the air is approximately $302.04\,\text{K}$.

6. Lecture 6 28/08/2024
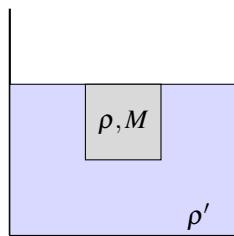
6.1 Archimedes Principle

Archimedes' principle states that any object completely or partially submerged in a fluid (liquid or gas) is buoyed up by a force equal to the weight of the fluid that the object displaces.

> **Upward force exerted by the fluid = weight of the fluid displaced by the object**

6.2 Buoyancy

Buoyancy is the upward force exerted by a fluid (liquid or gas) that opposes the weight of an object submerged in it.

Net force acting on mass(M) and density(ρ) submerged in fluid of density(ρ') is given by:

$$F_B = \rho'Vg - Mg \tag{76}$$
$$= \rho'Vg - \rho Vg \tag{77}$$
$$= (\rho' - \rho)Vg \tag{78}$$

Dividing equation with M on both side,

$$\frac{F_B}{M} = \frac{(\rho' - \rho)Vg}{M} \tag{79}$$
$$f_B = \frac{(\rho' - \rho)Vg}{\rho V} \tag{80}$$
$$= \left(\frac{\rho'}{\rho} - 1\right)g \tag{81}$$

If Buoyant force per unit mass (f_B),

$$f_B > 0 \rightarrow \text{upward force}$$
$$f_B < 0 \rightarrow \text{downward force}$$

We don't measure density in real case scenario, so we need to convert the equation in the useful form.
Assume pressure inside air parcel and surrounding equal and process to be reversible.
Using Ideal gas equation,

$$P = \rho R_d T_v \tag{82}$$
$$P = \rho' R_d T_v' \tag{83}$$

Substituting equation Eq.(82) & (83) in Eq.(81), we get:

$$f_B = \frac{\left(\frac{P}{R_d T_v'} - \frac{P}{R_d T_v}\right)}{\frac{P}{R_d T_v'}} g \tag{84}$$

$$\Rightarrow f_B = \frac{(T_v - T_v')}{T_v'} g \tag{85}$$

where T_v and T_v' are virtual temperture of of parcel and fluid respectively.

$$f_B > 0 \rightarrow T_v > T_v' \rightarrow \text{upward force}$$
$$f_B < 0 \rightarrow T_v < T_v' \rightarrow \text{downward force}$$
$$f_B = 0 \rightarrow T_v = T_v' \rightarrow \text{no net force}$$

> **Question 6.1:** A parcel ofair has a temperature of $29°C$ and specific humidity of $24g/kg$. It is embedded in the environment having termperature of $30°C$ and specific humidity of $5g/kg$
> **a.** What is vertical acceleration?
> **b.** If there are no forces acting on in, how long would take for thep parcel to raise $10m$ from starting position?

$\Rightarrow$ a. Given $T_{v,a} = 29°C = 302K$ and $T_{v,s} = 30°C = 303K$,
$q_a = R.H._{\cdot a} = 24g/kg$,
$q_s = R.H._{\cdot s} = 5g/kg$,
We know,

$$T_{v,a} = T_a(1 + 0.61q_a)$$
$$= 302(1 + 0.61 \times 24 \times 10^{-3})$$
$$= 306.42128K$$
$$T_{v,s} = T_s(1 + 0.61q_s)$$
$$= 303(1 + 0.61 \times 5 \times 10^{-3})$$
$$= 303.92415K$$

Buoyant force per unit mass f_B,

$$f_B = \left(\frac{T_{v,a} - T_{v,s}}{T_{v,s}}\right)g$$
$$= \left(\frac{306.42128 - 303.92415}{303.92415}\right) \times 9.81$$
$$= 0.0806m/s^2$$

$\therefore$ Vertical accerleration due to buoyant force is $0.0806m/s^2$.

$\Rightarrow$ b. Given Height $h = 10m$,
Vertical accerlation $a = f_B = 0.0806m/s^2$

Using equation of motion:
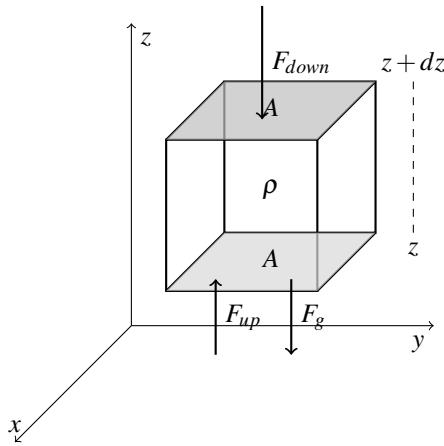
$$s = \frac{1}{2}at^2$$
$$10 = \frac{1}{2} \times 0.0806 \times t^2$$
$$t = 15.7524s$$

$\therefore$ Time taken by parcel to rasie $10m$ due to buoyant force is 15.75 seconds.

6.3 Hydrostatic equation

1. The downward force due to gravity:

$$F_g = mg = \rho(A\delta z)g \tag{86}$$

2. The upward force due to atmosphere force acting on the bottom of the slab is given by:

$$F_{up} = Ap(z) \tag{87}$$

3. The downward force acting on parcel:

$$F_{down} = Ap(z+dz) \tag{88}$$

$\therefore$ Net upward fore will be given by Eq.(90)

$$F = F_{up} - F_{down} - F_g \tag{89}$$
$$= Ap(z) - Ap(z+dz) - \rho(A\delta z)g \tag{90}$$

Hence, Upward acccleration will be:

$$a = \frac{F}{\rho A \delta z} \tag{91}$$

$$= \frac{Ap(z) - Ap(z+\delta z) - (A\delta z)\rho g}{\rho A \delta z} \tag{92}$$

$$= -\frac{1}{\rho}\left[\frac{p(z+\delta z) - p(z)}{\delta z}\right] - g \tag{93}$$

Taking $\lim_{\delta z \to 0}$

$$\Rightarrow a + g = -\frac{1}{\rho}\frac{dp}{dz} \tag{94}$$

Eq.(94) is called Hydrostatic equation.

> **Question 6.2: Velocity of hyricene is $10m/s$ and time taken $10min$ find acceleration.**
>
> $\Rightarrow$ Acceleration $a = \frac{v}{t} = \frac{10}{10\times 60} = \frac{10}{600} \approx 0.0167$

From the above example question 6.2 we can infer that $a + g \approx g$,

$\therefore$ we can rewrite Hydrostatic equation as following Eq.(95):

$$\Rightarrow g = -\frac{1}{\rho}\frac{dP}{dz} \tag{95}$$

This is called **Hydrostatication approximation**.

7. Lecture 7 29/08/2024

7.1 Ideal gas equation with Hydostatic equation
From Ideal gas equation:

$$P = \rho R_d T_v \tag{96}$$

$$\rho = \frac{P}{R_d T_v} \tag{97}$$

Substitute ρ in hydrostactic Eq.(95), we get:

$$g = -\frac{R_d T_v}{P}\frac{\partial P}{\partial z} \tag{98}$$

$$\frac{\partial P}{\partial z} = -\frac{P}{R_d T_v}g \tag{99}$$

$$\frac{1}{P}\frac{\partial P}{\partial z} = -\frac{g}{R_d T_v} \tag{100}$$

$$\frac{\partial \ln P}{\partial z} = -\frac{g}{R_d T_v} \tag{101}$$

The rate of change of logarithm of pressure is Inversely proportional to temperature and does not depend on pressure.

7.2 Geopotential
Geopotential at any point in the atmosohere is defined as the work done against the gravitational field to raise a mass of $1kg$ from sea level to that point,
Represnted by Eq.(102) and has unit Jkg^{-1}

$$d\phi = gdz \tag{102}$$

From hydrostatic equation Eq.(94),

$$dp = -\rho gdz \tag{103}$$

$$gdz = -\frac{1}{\rho}dp \tag{104}$$

$$gdz = -\alpha dp \tag{105}$$

where α is specific volume.
Intergrating Eq.(102), we get:

$$\int_0^{\phi(z)} d\phi = \int_0^z gdz \tag{106}$$

$$\phi(z) = \int_0^z gdz \tag{107}$$

Let g_0 be accerleration due to gravity averaged over the surface.

$$\frac{\phi(z)}{g_0} = \int_0^z \frac{g}{g_0}dz \tag{108}$$

$$\Rightarrow Z = \frac{\phi(z)}{g_0} \tag{109}$$

The Z **in Eq.(109) is called Geopotential height**.

$$p = \rho R_d T_v \tag{110}$$

$$p\alpha = R_d T_v \tag{111}$$

$$\alpha = \frac{R_d T_v}{p} \tag{112}$$

z (km)	Z (km)	g (m/s^2)
0	0	9.81
1	1	9.80
10	9.99	9.77
100	98.87	9.50
500	46.36	8.43

Table 2. Deviation of vaules of g for Geometric Height(z), Geopotential Height(Z)

$$d\phi = gdz = -\alpha dp \tag{113}$$

$$d\phi = -\frac{R_d T_v}{p}dp \tag{114}$$

Integrating from both sides, we get:

$$\int_{\phi_1}^{\phi_2} d\phi = \int_{P_1}^{P_2} -\frac{R_d T_v}{p}dp \tag{115}$$

$$\phi_2 - \phi_1 = -R_d \int_{P_1}^{P_2} T_v \frac{dp}{p} \tag{116}$$

Dividing both side with g_0, we get:

$$\frac{(\phi_2 - \phi_1)}{g_0} = -\frac{R_d}{g_0}\int_{P_1}^{P_2} T_v \frac{dp}{p} \tag{117}$$

$$(Z_2 - Z_1) = -\frac{R_d}{g_0}\int_{P_1}^{P_2} T_v \frac{dp}{p} \tag{118}$$

By assuming isothermal atmosphere

$$(Z_2 - Z_1) = -\frac{R_d}{g_0}\int_{P_1}^{P_2} \bar{T}_v \frac{dp}{p} \tag{119}$$

$$(Z_2 - Z_1) = -\frac{R_d}{g_0}\bar{T}_v \ln\left(\frac{dp}{p}\right) \tag{120}$$

$$\Rightarrow (Z_2 - Z_1) = -H \ln\left(\frac{dp}{p}\right) \tag{121}$$

Where $\bar{T}_v$ is average temperature of atmosphere taken over geopotnetial ($\phi_1 \& \phi_2$) and H is scale height giev by Eq.(122):

$$H = \frac{R_d T_v}{g_0} \tag{122}$$

Scale height H is defined as height at which the pressure reduces to $1/e$ times the surface pressure. It is around $7.8km$ for Earth's atmosphere.
Eq.(121) is called Hypsometric equation.
Simplifing Eq.(121), we get:

$$P_2 = P_1 e^{-\frac{(Z_2 - Z_1)}{H}} \tag{123}$$

$$\Rightarrow P = P_0 e^{-\frac{(Z_2 - Z_1)}{H}} \tag{124}$$

Question 7.1:

$\Rightarrow$ Solution,

$$
\begin{aligned}
(Z_2 - Z_1) &= -\frac{R_d}{g_0}\bar{T_v}\ln\left(\frac{dp}{p}\right) \\
&= -\frac{287 \times 255}{9.81}\ln\frac{P_2}{P_1} \\
&= -7.4\ln\frac{P_2}{P_1}
\end{aligned}
$$

8. Lecture 8 30/08/2024

Question 8.1: On May 20, 2020 tropical cyclone Ampan of centre pressure at ocean surface dropped to $920hPa$. The surrounding region away from influence of centre of cyclone had mean sea level pressure of $1010hPa$. The height depression associated with centre of cyclone vanished at height of pressure level of $150hPa$. If the mean virtual temperature of the surrounding between the surface at $150hPa$ was $-10°C$. What was the corresponding mean virtual temperature in the centre of storm?

$\Rightarrow$ Given data:

$$P_1 = 1010hPa = 101000Pa$$
$$P_2 = 920hPa = 92000Pa$$
$$P_3 = 150hPa = 15000Pa$$
$$T_{v,surr} = -10°C = 263.15K$$
$$R_d = 287J/(kgK)$$
$$g_0 = 9.81m/s^2$$

$\Rightarrow$ Height difference calculation in the surrounding air:

$$(Z_2 - Z_1) = \frac{R_d T_{v,surr}}{g_0} \ln\left(\frac{P_1}{P_3}\right)$$
$$= \frac{287 \times 263.15}{9.81} \ln\left(\frac{101000}{15000}\right)$$
$$= 7701.43 \times 1.906$$
$$= 14681.92\,\text{m}$$

$\Rightarrow$ Solution for mean virtual temperature at the center of the storm:

$$(Z_2 - Z_1) = -\frac{R_d}{g_0} \bar{T}_{v,center} \ln\left(\frac{P_3}{P_2}\right)$$
$$14681.92 = -\frac{287}{9.81} \bar{T}_{v,center} \ln\left(\frac{15000}{92000}\right)$$
$$\bar{T}_{v,center} = -\frac{14681.92 \times 9.81}{287\ln\left(\frac{15000}{92000}\right)}$$
$$= -\frac{14681.92 \times 9.81}{287 \times (-1.7749)}$$
$$= \frac{144040.6}{-509.4613}$$
$$= 282.72\,\text{K}$$
$$T_{v,center} \approx 282.72 - 273.15$$
$$T_{v,center} \approx 9.57°C$$

Question 8.2: Calculate the thickness of layer between $1000hPa$ and $500hPa$ pressure surface.
a. At point in tropics where T_v is $15°C$
b. At point in polar where T_v is $-40°C$

Solution $\Rightarrow$ a.

$$(Z_2 - Z_1) = -\frac{R_d}{g_0} \bar{T}_v \ln\left(\frac{dp}{p}\right)$$
$$= \frac{287 \times 288K}{9.81m/s^2} \ln\frac{1000hPa}{500hPa}$$
$$= 5840.2419km$$

$\Rightarrow$ b.

$$(Z_2 - Z_1) = -\frac{R_d}{g_0} \bar{T}_v \ln\left(\frac{dp}{p}\right)$$
$$= \frac{287 \times 233K}{9.81m/s^2} \ln\frac{1000hPa}{500hPa}$$
$$= 4724.9179km$$

Pressure profiles in the idealized atmosphere
8.1 Constant density atmosphere

Assume atmosphere is at hydrostatic balance and density ρ to be constant.

$$dP = -\rho g dz \tag{125}$$
$$\int_{P(0)}^{P(z)} dP = -\int_0^z \rho g dz \tag{126}$$
$$p(z) - p(0) = -\rho g z \tag{127}$$

Substituting values in above eqaution, we obtain:

$$(0 - 101.3) = -1.23 \times 9.8 \times z \tag{128}$$
$$z \approx 8.3952km \tag{129}$$

Using Ideal gas equation, substitute P,

$$d(\rho RT) = -\rho g dz \tag{130}$$
$$RdT = -gdz \tag{131}$$
$$dT = -\frac{g}{R}dz \tag{132}$$
$$T(z) - T(0) = \frac{g}{R}(z - 0) \tag{133}$$
$$T(z) = T(0) - \frac{g}{R}z \tag{134}$$
$$T(z) = T(0) - 0.0341z \tag{135}$$

$$T(z) = T(0) - 34.1z \tag{136}$$

where 34.1 constant have an unit of $°C/km$

$$\Rightarrow \Gamma = -\frac{dT}{dz} = -\frac{g}{R} = -34.1°C/km \tag{137}$$

This is Ideal/theoritical value, but **actual/pratical value for** Γ **is** $6.5°/km$ because of phenomenon called **auto-convective lapse rate** i.e. ρ varies with altitude, and warm are and cold air do vertical circulation.

8.2 Constant temperature atmosphere

Assume atmosphere is at hydrostatic balance and Temperature T to be constant.

$$dP = -\rho g dz \tag{138}$$

$$d\rho RT = -\rho g dz \tag{139}$$

$$p(z) - p(0) = -\rho g dz \tag{140}$$

$$RT d\rho = -\rho g dz \tag{141}$$

$$\frac{d\rho}{\rho} = -\frac{g}{RT} dz \tag{142}$$

$$\ln \rho |_{\rho_1}^{\rho_2} = -\frac{g}{RT} z |_0^z \tag{143}$$

$$\Rightarrow \ln \frac{\rho_2}{\rho_1} = -\frac{g}{RT} z \tag{144}$$

8.3 Constant lapse rate atmosphere

$$T = T_0 - \Gamma_z \tag{145}$$

$$dP = -\rho g dz \tag{146}$$

$$\frac{dP}{dz} = -\frac{Pg}{RT} \tag{147}$$

$$\frac{dP}{dz} = -\frac{Pg}{R(T_0 - \Gamma_z)} \tag{148}$$

$$\frac{1}{P} dP = -\frac{g dz}{R(T_0 - \Gamma_z)} \tag{149}$$

$$\int_{P_1}^{P_2} \frac{1}{P} dP = -\frac{g}{R} \int_0^z \frac{dz}{(T_0 - \Gamma_z)} \tag{150}$$

$$\Rightarrow \ln \frac{P_2}{P_1} = -\frac{g}{R\Gamma_z} \ln \left(\frac{T_0 - \Gamma_z}{T_0} \right) \tag{151}$$

9. Lecture 9 04/09/2024

9.1 1st law of thermodynamics and it's application

Pressure-volume work

Work done δw by any force F to displace object with diplacement ds is equal to:

$$\delta w = F \cdot ds \tag{152}$$
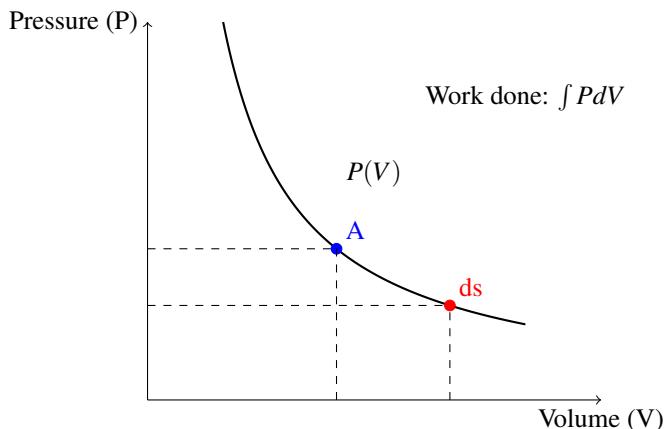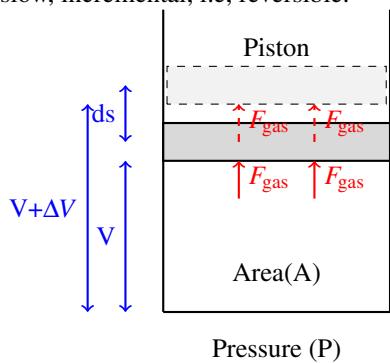
Incremental Work done by Force F to increase the volume will be given as follows:

$$\delta w = F \cdot ds \tag{153}$$
$$= PA \cdot ds \tag{154}$$
$$= PdV \tag{155}$$

Assuming Pressure P constant at each step, and process is slow, incremental, i.e, reversible.

10. Lecture 10 05/09/2024

10.1 . 1st law of thermodynamics and it's application

$$\Delta U = Q - W \tag{156}$$

Mechanical work

$$\delta w = F \cdot ds \tag{157}$$

Pressure-volume work

$$\delta w = PA \cdot ds \tag{158}$$

$$\delta w = PdV \tag{159}$$

$$w = \int_i^f PdV \tag{160}$$

P (Pressure)

$$w = \int_i^f PdV$$

V (Volume)

$$w = \int_i^f PdV \tag{161}$$

$$w = \int_i^f PdV + \int_f^i PdV \tag{162}$$

$$w = \left[\int_i^f PdV \right]_1 - \left[\int_i^f PdV \right]_2 \tag{163}$$

$$w \neq 0 \tag{164}$$

$$\Rightarrow \oint_C w = \oint_C Pdv \neq 0 \tag{165}$$
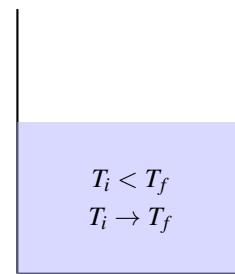
$$w = \int_i^f PdV \tag{166}$$

$$w = \int_i^f PdV + \int_f^i PdV \tag{167}$$

$$w = \left[\int_i^f PdV \right]_1 - \left[\int_i^f PdV \right]_2 \tag{168}$$

$$w = 0 \tag{169}$$

$$\Rightarrow \oint_C w = \oint_C Pdv = 0 \tag{170}$$

$$\delta w = PAds \tag{171}$$

$$\delta w = F \cdot ds \tag{172}$$

$$\frac{\delta w}{dt} = m \frac{dv}{dt} \frac{ds}{dt} \tag{173}$$

$$\frac{\delta w}{dt} = mv \frac{dv}{dt} \tag{174}$$

$$\frac{\delta w}{dt} = \frac{d}{dt} \left(\frac{1}{2} mv^2 \right) \tag{175}$$

$$\Rightarrow \frac{\delta w}{dt} = \frac{d}{dt} (K.E.) \tag{176}$$

Formulation of 1st law of thermodynamics

Case 1: Heating

$$\Delta U = Q$$

Case 2: By doing work

$$\Delta U = -W$$

From case 1 and 2 for we can write 1st law of thermodynamics as:

$$\Delta U = Q - W \tag{177}$$

$$\delta q = du + \delta w \tag{178}$$

$$T_i < T_f$$
$$T_i \to T_f$$

↑ Heat Δ ↑

In terms of Intensive parameters

$$\delta q = du + pd\alpha \tag{179}$$

1. Pressuer-volume wok done by a system = reduction in internal energy + heat supplied by the environment.

2. Pressuer-volume wok done on a system = increase in internal energy + heat transfered to the environment.

10.2 . Heat capacity

$$\frac{\delta q}{dT} = C \tag{180}$$

Unit $Jk^{-1}Kg^{-1}$

Ideal gas equation:

$$P\alpha = R_d T$$

Case 1: Increase in volume (If pressure is kept constant → Isobaric process)

Case 2: Increase in pressure (If volume is kept constant → Isochoric process)

Case 3: Combination of above the both cases. (i.e. increase in both pressure and volume)

10.3 . Heat capacity at constant volume

From Eq.(179)

$$\delta q = du + pd\alpha \tag{181}$$

$$\delta q = du \tag{182}$$

$$\delta q = C_v dT \tag{183}$$

$$C_v = \left(\frac{\delta q}{dT}\right)_{\alpha=\text{cont}} \tag{184}$$

$$C_v = \left(\frac{du}{dT}\right)_{\alpha=\text{cont}} \tag{185}$$

$$\Rightarrow du = C_v dt \tag{186}$$

From Kinetic theory of gas

$$U = \frac{3}{2}P\alpha = \frac{3}{2}R_d T$$

For monoatomic gas:

$$C_v = \frac{du}{dt} = \frac{3}{2}R_d = 430.5 Jk^{-1}kg^{-1}$$

For diatomic gas:

$$C_v = \frac{du}{dt} = \frac{5}{2}R_d = 718 Jk^{-1}kg^{-1}$$

11. Lecture 11 06/09/2024

11.1 . Specific heat capacity

Specific heat = $\frac{\delta q}{dT}$

$$\delta q = du + \delta w \tag{187}$$

$$\delta q = du + pd\alpha \tag{188}$$

11.2 . Specific heat at constant volume

Volume(V) is constant, i.e., specific density(α) = constant
$\therefore d\alpha = 0$

$$\delta q = du + pd\alpha \tag{189}$$

$$\delta q = du \tag{190}$$

$$\delta q = C_v dT \tag{191}$$

$$C_v = \left(\frac{\delta q}{dT}\right)_{\alpha=\text{cont}} \tag{192}$$

$$C_v = \left(\frac{du}{dT}\right)_{\alpha=\text{cont}} \tag{193}$$

$$du = C_v dt \tag{194}$$

Hence,

$$\Rightarrow \delta q = C_v dt + Pd\alpha \tag{195}$$

$$C_v = \begin{cases} \frac{3}{2}R_d & \text{for monoatomic gas,} \\ \frac{5}{2}R_d & \text{for diatomic gas.} \end{cases}$$

Rotational K.E. is significant for diaatomic gas but not for monoatomic gas.

Monoatomic Gas   Diatomic Gas

Diatomic Gas

11.3 . Specific heat at constant pressure

Pressure(P) is constant $\therefore dP = 0$
From Ideal gas eqaution

$$P = R_d T \tag{196}$$

$$dP\alpha = Pd\alpha + \alpha dP \tag{197}$$

$$Pd\alpha + \alpha dP = d(R_d T) \tag{198}$$

$$Pd\alpha + \alpha dP = R_d dT \tag{199}$$

$$Pd\alpha = R_d dT - \alpha dP \tag{200}$$

$$\delta q = du + pd\alpha \tag{201}$$

$$\delta q = du \tag{202}$$

$$\delta q = C_v dT \tag{203}$$

$$C_v = \left(\frac{\delta q}{dT}\right)_{\alpha=\text{cont}} \tag{204}$$

$$C_v = \left(\frac{du}{dT}\right)_{\alpha=\text{cont}} \tag{205}$$

$$du = C_v dt \tag{206}$$

From Eq.(195) and Eq.(200)

$$\delta q = C_v dT + R_d dT - \alpha dP \tag{207}$$

$$\delta q = (C_v + R_d)dT - \alpha dP \tag{208}$$

$$\Rightarrow \delta q = C_p dT - \alpha dP \tag{209}$$
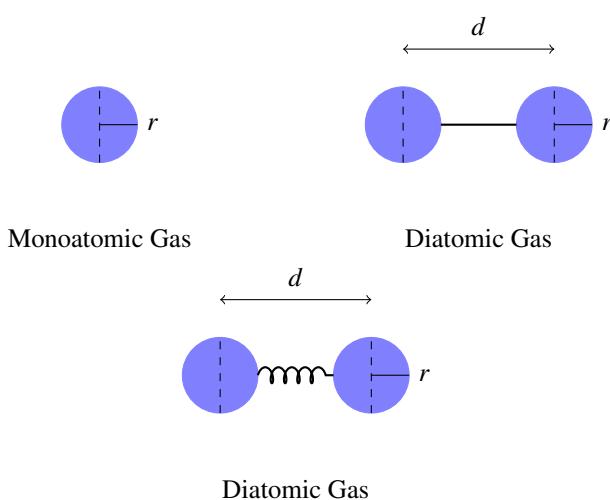
$$C_p = \begin{cases} 717.5\,\text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1} & \text{for monoatomic gas} \\ 1005\,\text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1} & \text{for diatomic gas} \end{cases}$$

11.4 . Special forms of 1st law of thermodynamics
I. Isobaric process

$$\delta q = C_p dT \tag{210}$$

$$= \left(\frac{C_p}{C_v}\right)C_v dT \tag{211}$$

$$= \left(\frac{C_p}{C_v}\right)dU \tag{212}$$

$$= \gamma dU \tag{213}$$

II. Isothermal process

$$\delta q = -\alpha dP = Pd\alpha = \delta w \tag{214}$$

$$q = \int_i^f \alpha dP \tag{215}$$

$$= \int_i^f \frac{R_d T}{P}dP \tag{216}$$

$$= -R_d T \ln P|_i^f \tag{217}$$

$$= R_d T \ln \alpha|_i^f = w \tag{218}$$

III. Isochoric process

$$\delta q = C_v dT = dud\alpha = 0 \tag{219}$$

$$q = \int_i^f C_v dT \tag{220}$$

$$= C_v(T_f - T_i) \tag{221}$$

$$= C_v\Delta T = U \tag{222}$$

IV. Adiabatic process

$$0 = C_v dT + P\alpha \tag{223}$$

$$0 = C_p dT - P\alpha \tag{224}$$

Question 11.1: For each of the following conditions compute:
i. Mechanical work done by the sample of air.
ii. Heat added to the sample.
a. Isothermal cpmpression to $1/5^{th}$ **of it's original volume at** $15°C$**.**
b. Isobaric heating from $0°C$ **to** $20°C$**.**
c. Adiabatic expansion to 5 **times it's orignanal volume within initial temperature of** $20°C$**.**

$\Rightarrow$ Solution:

a. For Isothermal process

$$q = w = R_d T \ln \left(\frac{1}{V}\right)\Big|_V^{V/5}$$
$$= 287 \times 288 \times T \ln(5)$$
$$= 133.0297 kJ$$

b. For Isobaric heating

$$q = C_p dT$$
$$= \frac{7}{2} R_d \times \Delta T$$
$$= \frac{7}{2} \times 287 \times 20$$
$$= 19740 J$$
$$\approx 20 kJ$$

c. Adiabatic heating

$$w = \frac{P_1 V_1 - P_2 V_2}{\gamma - 1}$$
$$= \frac{R_d (T_1 - T_2)}{\gamma - 1}$$
$$= \frac{287 \times (293 - \frac{293}{5})}{\frac{7}{5} - 1}$$
$$= 168.182 kJ$$
$$q = 0 J$$

12. Lecture 12 11/09/2024

12.1 . Poisson's equation for adiabatic transformation

For adiabatic processes:

$$dq = 0 \tag{225}$$

$$c_p dT = \alpha dP \tag{226}$$

$$c_p \frac{dT}{T} = \alpha \frac{dP}{T} \tag{227}$$

From Ideal gas equations:

$$P\alpha = R_d T \tag{228}$$

$$\alpha = \frac{R_d T}{P} \tag{229}$$

$$c_p \frac{dT}{T} = \frac{R_d T}{P} \frac{dP}{T} \tag{230}$$

$$c_p \frac{dT}{T} = R_d \frac{dP}{P} \tag{231}$$

Integrating from an initial termerature T_0 and pressure P_0 to arbitary temperature and pressure T and P, we get:

$$\int_{T_0}^{T} c_p \frac{dT}{T} = \int_{P_0}^{P} R_d \frac{dP}{P} \tag{232}$$

$$c_p \ln\left(\frac{T}{T_0}\right) = R_d \ln\left(\frac{P}{P_0}\right) \tag{233}$$
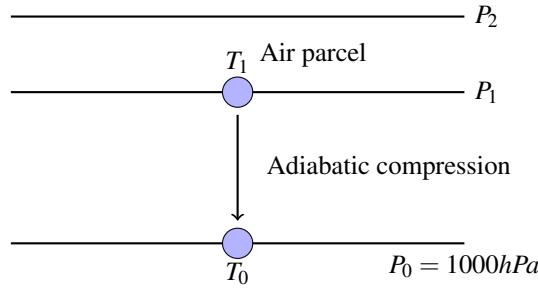
$$\left(\frac{T}{T_0}\right)^{c_p} = \left(\frac{P}{P_0}\right)^{R_d} \tag{234}$$

$$T_0 = T\left(\frac{P}{P_0}\right)^{\frac{R_d}{c_p}} \tag{235}$$

$$\Rightarrow \theta = T\left(\frac{1000}{P_0}\right)^{k} \tag{236}$$

where constant k is R_d/c_p which is equal to **0.286**, $P_0 = 1000hPa$ which is **near surface pressure** and θ is known as **potential temperature**.

Potential temperature$()\theta$ in Eq.(236) is defined as a adiabaticty compressed parcel is bought to $1000hPa$ isobar (near surface) from that leveland temperature is measure.

Let us assume that

$$\theta = ATP^{-k} \tag{237}$$

where k is $R_d/C_p = 0.286$.

Taking logarithm on both sides

$$d(\ln\theta) = d(\ln A) + d(\ln T) - kd(\ln P) \tag{238}$$

$$d(\ln\theta) = d(\ln T) - \frac{R_d}{C_p}d(\ln P) \tag{239}$$

where $A = (P_0)^k = (1000)^{0.286}$

Consider 1st law of thermodynamics

$$\delta q = C_p dT - \alpha dP \tag{240}$$

$$\frac{\delta q}{T} = C_p \frac{dT}{T} - \alpha \frac{dP}{T} \tag{241}$$

$$\frac{\delta q}{C_p T} = \frac{dT}{T} - \frac{R_d T dP}{C_p P T} \tag{242}$$

$$\frac{\delta q}{C_p T} = \frac{dT}{T} - \frac{R_d dP}{C_p P} \tag{243}$$

$$\frac{\delta q}{C_p T} = d(\ln T) - \frac{R_d}{C_p}d(\ln P) \tag{244}$$

from Eq.(239) and Eq.(244)

$$d(\ln\theta) = \frac{\delta q}{C_p T} \tag{245}$$

For adiabatic process $\delta q = 0$

$$d(\ln\theta) = 0 \tag{246}$$

θ is constant, conserved for adiabatic process.

Question 12.1: **Transcontinental airline fly at an altitude of** $12km$ **where the temperature outside is** $-55°C$ **and the pressure is approxiately** $200hPa$
a. Compute the potential temperature of air at this altitude.
b. Cabin pressure is typically mentioned of $750hPa$ **corresponding to pressure alttitude of** $2.24km$**. when outside air is adiabatically compressed to cabin pressure. Compute the air temperature if no corrective operation where taken.**

$\Rightarrow$ Solution:
a. Potential temperature of air

$$\theta = T\left(\frac{1000}{P}\right)^{k}$$
$$= 218\left(\frac{1000}{200}\right)^{0.286}$$
$$= 218(5)^{0.286}$$
$$= 345.4317K$$
$$= 72.43°C$$

b. Potential temperature of air

$$\theta = T\left(\frac{1000}{P}\right)^{k}$$
$$= 218\left(\frac{750}{200}\right)^{0.286}$$
$$= 218\left(\frac{4}{3}\right)^{0.286}$$
$$= 318.14K$$
$$= 45°C$$

13. Lecture 13 12/09/2024

13.1 . Adiabatic transformation - Poisson's equation

1. Case I.

$$C_p dT = \alpha dP \tag{247}$$

$$C_p \frac{dT}{T} = R_d \frac{dP}{P} \tag{248}$$

$$\frac{dT}{T} = \left(\frac{C_p - C_v}{C_p}\right)\frac{dP}{P} \tag{249}$$

$$\frac{dT}{T} = \left(1 - \frac{C_v}{C_p}\right)\frac{dP}{P} \tag{250}$$

$$\frac{dT}{T} = \left(1 - \frac{1}{\gamma}\right)\frac{dP}{P} \tag{251}$$

$$\frac{dT}{T} = \left(\frac{\gamma - 1}{\gamma}\right)\frac{dP}{P} \tag{252}$$

$$\ln T = \left(\frac{\gamma - 1}{\gamma}\right)\ln P + \ln C \tag{253}$$

$$T = CP^{\left(\frac{\gamma - 1}{\gamma}\right)} \tag{254}$$

$$TP^{\left(\frac{1-\gamma}{\gamma}\right)} = C \tag{255}$$

2. Case II.

$$C_v dT = -P d\alpha \tag{256}$$

$$C_v \frac{dT}{T} = -R_d \frac{d\alpha}{\alpha} \tag{257}$$

$$\frac{dT}{T} = -\left(\frac{C_p - C_v}{C_v}\right)\frac{d\alpha}{\alpha} \tag{258}$$

$$\frac{dT}{T} = \left(1 - \frac{C_p}{C_v}\right)\frac{d\alpha}{\alpha} \tag{259}$$

$$\frac{dT}{T} = \left(1 - \gamma\right)\frac{d\alpha}{\alpha} \tag{260}$$

$$\ln T = (1 - \gamma)\ln \alpha + \ln C \tag{261}$$
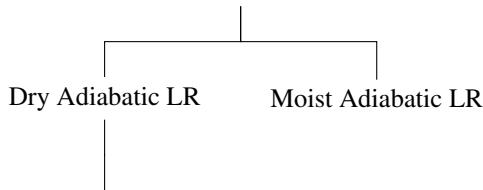
$$\ln T = \ln \alpha^{(1-\gamma)} + \ln C \tag{262}$$

$$T = C\alpha^{(1-\gamma)} \tag{263}$$

$$T\alpha^{(\gamma - 1)} = C \tag{264}$$

13.2 . Adiabatic Lapse Rate

Adiabatic Lapse Rate (LR)

Dry Adiabatic LR   Moist Adiabatic LR

Phase change of moisture is absent

Note: Moisture is present in dry air parcel but it is assumed that it does not show any phase chage.

13.3 . Dry Adiabatic Lapse Rate (DALR)

Using the 1st law of thermodynamics

$$\delta q = C_p dT - \alpha dP \tag{265}$$

Adiabatic process:

$$\delta q = 0 \tag{266}$$

$$C_p dT = \alpha dP \tag{267}$$

Lapse rate:

$$\frac{dT}{dz} = \frac{dT}{dP} \cdot \frac{dP}{dz} \tag{268}$$

From Eq.(267) and (268)

$$\frac{dT}{dz} = \frac{R_d T}{C_p P} \cdot \frac{dP}{dz} \tag{269}$$

Note: Hydrostatic balance is applied to surrounding not on parcel.

Appling hydrostatic approximaion, the pressure of unconfined air pacel is same as that of the envirnment (i.e. $P = P'$), we get:

$$\frac{dP}{dz} = \frac{dP'}{dz} = \rho' g = -\frac{P' g}{R_d T'} \tag{270}$$

where P' and T' are ambient pressure and temperature.

From Eq.(267), Eq.(268) and Eq.(270)

$$\frac{dT}{dz} = \frac{R_d T}{C_p P} \cdot \left(-\frac{P' g}{R_d T'}\right) \tag{271}$$

$$\frac{dT}{dz} = \frac{g}{C_p} \cdot \left(-\frac{T P'}{T' P}\right) \tag{272}$$

$$\frac{dT}{dz} = -\frac{g}{C_p}\left(\frac{T}{T'}\right) \tag{273}$$

$$\Gamma_d = -\frac{dT}{dz} = \frac{g}{C_p}\left(\frac{T}{T'}\right) \tag{274}$$

$$\Gamma_d = -\frac{dT}{dz} \approx \frac{g}{C_p} \tag{275}$$

$$\Gamma_d = -\frac{dT}{dz} \approx 9.8 \times 10^{-3}\,^\circ C/m \tag{276}$$

$$\Rightarrow \Gamma_d = -\frac{dT}{dz} \approx 9.8\,^\circ C/km \tag{277}$$
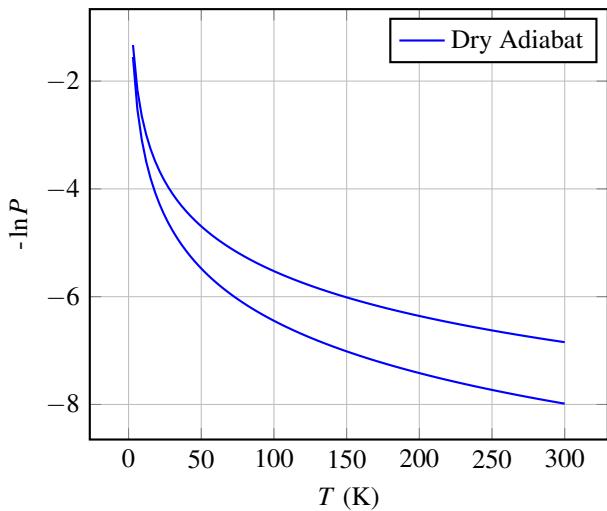
where Γ_d is **Dry Adiabatic Lapse Rate (DALR)** which is aproximately equal to $9.8\,^\circ C/km$, also T and T' are comparably equal.

13.4 . Possion's equations

From Eq.(236)

$$\theta = T\left(\frac{1000}{P}\right)^k \tag{278}$$

$$\Rightarrow P^k = \left(\frac{P_0^k}{\theta}\right)T \tag{279}$$

14. Lecture 14 20/09/2024

14.1 . Heat Engines

1. Construct closed cycle of compression and expansion to produce net work.

2. Production of work required expenditure of internal energy on heat supplied by the environment.

From 1st law of thermodynamics

$$\delta q = du + \delta w \tag{280}$$

$$\oint \delta q = \oint du + \oint \delta w \tag{281}$$

$$\oint \delta q = \oint \overset{0}{du} + \oint \delta w \quad (\because u_i = u_f) \tag{282}$$

$$\oint \delta q = \oint \delta w \tag{283}$$

$$\Rightarrow q_{\text{net}} = w_{\text{net}} \quad \text{(Theoritically)} \tag{284}$$

In theory, the equality $q_{\text{net}} = w_{\text{net}}$ arises from the assumption of a perfect heat engine operating in a closed cycle. This idealized scenario implies that all the heat energy supplied to the engine is converted into mechanical work without any losses due to friction, heat dissipation, or other irreversibilities. However, no physical engine can achieve this ideal efficiency. Real engines inevitably encounter energy losses through various mechanisms, such as:

1. **Heat Loss:** Part of the input heat is lost to the surroundings, decreasing the effective energy available for work.

2. **Friction:** Mechanical losses due to friction in moving parts result in energy dissipation as heat, further reducing the net work output.

3. **Non-ideal Processes:** Real thermodynamic processes often involve irreversible changes, which lead to additional energy losses that are not accounted for in the ideal model.

Thus, while $q_{\text{net}} = w_{\text{net}}$ serves as a theoretical benchmark, practical engines operate with efficiencies below this ideal, governed by real-world constraints and inefficiencies.

14.2 . Efficiency

Efficiency of heat engine,

$$\eta = \frac{q_{\text{in}} - q_{\text{out}}}{q_{\text{in}}} = \frac{w}{q_{\text{in}}} \tag{285}$$

14.3 . Carnot Cycle

The Carnot cycle consists of the following four steps, whcih can be seen in figure:

1. **Step 1:** Reversible isothermal expansion.
2. **Step 2:** Reversible adiabatic expansion.
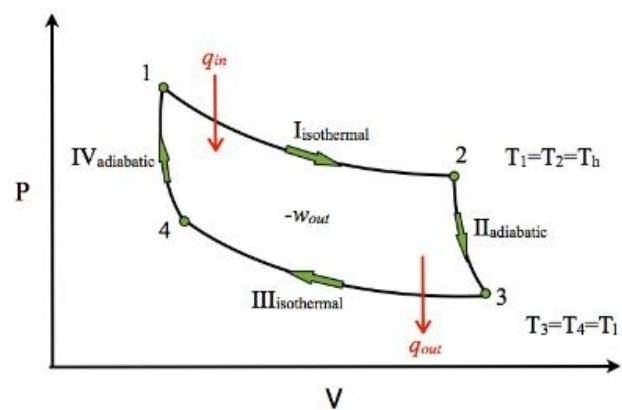
Figure 4. Carnot Cycle

3. **Step 3:** Reversible isothermal compression.
4. **Step 4:** Reversible adiabatic compression.

Step 1: Reversible Isothermal Expansion
During this process, the temperature remains constant, i.e., $\Delta T = 0$, which implies that the internal energy change is zero, $\Delta u = 0$.
The work done during this process is given by:

$$w_{12} = \int \delta w = \int_{\alpha_1}^{\alpha_2} p \, d\alpha \tag{286}$$

$$= \int_{\alpha_1}^{\alpha_2} \frac{R_d T_1}{\alpha} \, d\alpha \tag{287}$$

$$= R_d T_1 \ln \frac{\alpha_2}{\alpha_1} \tag{288}$$

Since $du = 0$ (because $dT = 0$), we have:

$$du_{12} = dw_{12} \quad \Rightarrow \quad Q_{12} = W_{12}$$

Step 2: Reversible Adiabatic Expansion
In this step, the gas expands without heat exchange, leading to a change in internal energy. The change in internal energy is given by:

$$\Delta u_{23} = C_v(T_2 - T_1) \tag{289}$$

$$-\Delta u_{23} = w_{23} \tag{290}$$

$$w_{23} = -C_v(T_2 - T_1) \tag{291}$$

Step 3: Reversible Isothermal Compression
In this process, the gas is compressed isothermally, maintaining a constant temperature:

$$w_{34} = \int \delta w = \int_{\alpha_3}^{\alpha_4} p \, d\alpha \tag{292}$$

$$= \int_{\alpha_3}^{\alpha_4} \frac{R_d T_2}{\alpha} \, d\alpha \tag{293}$$

$$= R_d T_2 \ln \frac{\alpha_4}{\alpha_3} \tag{294}$$

Step 4: Reversible Adiabatic Compression
During this step, the gas is compressed adiabatically, resulting in a temperature increase and work done on the gas. The work can be expressed as:

$$w_{41} = C_v(T_2 - T_1) \tag{295}$$

From Poisson's equation (236), we have the following relations for the Carnot cycle:

$$T_1 \alpha_2^{\gamma-1} = T_2 \alpha_3^{\gamma-1} \tag{296}$$

$$\frac{T_1}{T_2} = \left(\frac{\alpha_3}{\alpha_2}\right)^{\gamma-1} \tag{297}$$

Similarly, for the other isentropic process:

$$T_1 \alpha_1^{\gamma-1} = T_2 \alpha_4^{\gamma-1} \tag{298}$$

$$\frac{T_1}{T_2} = \left(\frac{\alpha_4}{\alpha_1}\right)^{\gamma-1} \tag{299}$$

From these two equations, we can equate the temperature ratios:

$$\left(\frac{\alpha_3}{\alpha_2}\right)^{\gamma-1} = \left(\frac{\alpha_4}{\alpha_1}\right)^{\gamma-1} \tag{300}$$

$$\frac{\alpha_3}{\alpha_4} = \frac{\alpha_2}{\alpha_1} \tag{301}$$

Total Work:
From Eq.(288), (291), (294), and (295), the total work done in the cycle is:

$$W_{\text{Total}} = w_{12} + w_{23} + w_{34} + w_{41} \tag{302}$$

$$= R_d T_1 \ln\frac{\alpha_2}{\alpha_1} - C_v(T_2 - T_1) + \tag{303}$$

$$R_d T_2 \ln\frac{\alpha_4}{\alpha_3} + C_v(T_2 - T_1) \tag{304}$$

$$= R_d T_1 \ln\frac{\alpha_2}{\alpha_1} + R_d T_2 \ln\frac{\alpha_4}{\alpha_3} \tag{305}$$

$$= R_d \left[T_1 \ln\frac{\alpha_2}{\alpha_1} + T_2 \ln\frac{\alpha_4}{\alpha_3}\right] \tag{306}$$

$$= R_d(T_1 - T_2)\ln\left(\frac{\alpha_2}{\alpha_1}\right) \quad \text{From Eq.(301)} \tag{307}$$

Finding the effiency of carnot cycle using Eq.(285) and Eq.(307)

$$\eta = \frac{Q_1 + Q_2}{Q_1} \tag{308}$$

$$= 1 + \frac{Q_2}{Q_1} \tag{309}$$

$$= 1 + \frac{R_d T_2 \ln\frac{\alpha_3}{\alpha_4}}{R_d T_1 \ln\frac{\alpha_2}{\alpha_1}} \tag{310}$$

$$= 1 - \frac{T_2 \ln\frac{\alpha_2}{\alpha_1}}{T_1 \ln\frac{\alpha_2}{\alpha_1}} \tag{311}$$

$$\Rightarrow \eta = 1 - \frac{T_2}{T_1} \tag{312}$$

The temperature at which carnot engine becomes 100% efficient (i.e. $\eta = 1$) is called **Absolute temeperature** ($0K$)
In real world scenarios $0K$ is not acheivable and there is no process which is ideally reversible.
//

> **Question 14.1: For a potential temperature of $290K$ compute the corresponding temperature at $700hPa$ and $500hPa$. Sketch the corresponding adiabat in skew-T diagram.**
>
> $$\Rightarrow \theta = T\left(\frac{1000}{P_0}\right)^k \quad \text{, where } k = \frac{R_d}{C_p} = \frac{287}{1004} = 0.286$$
> $$T = \theta\left(\frac{P_0}{1000}\right)^k$$
> $$T_1 = \theta\left(\frac{700}{1000}\right)^{0.286} = 261.87K \approx -11.2°C$$
> $$T_2 = \theta\left(\frac{500}{1000}\right)^{0.286} = 237.88K \approx -35.15°C$$

14.4 . Skew T - log P diagram
Skew-T plot[1] shows combined and also seperate plots for Full Skew-T, Isothermal lines, Isobars, Dry Adiabats, Moist Adiabats, Mixing Ratio and Wind Staff.

[1] https://www.noaa.gov/jetstream/upperair/skew-t-log-p-diagrams
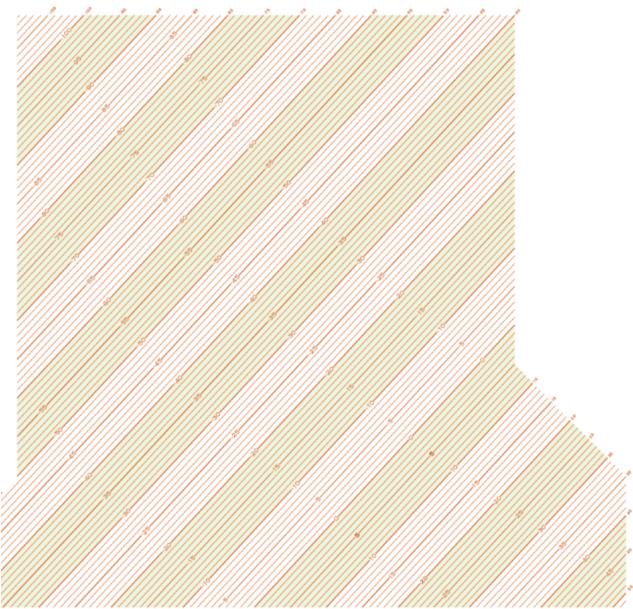
**Figure 5.** Isothermal lines in Skew-T diagram
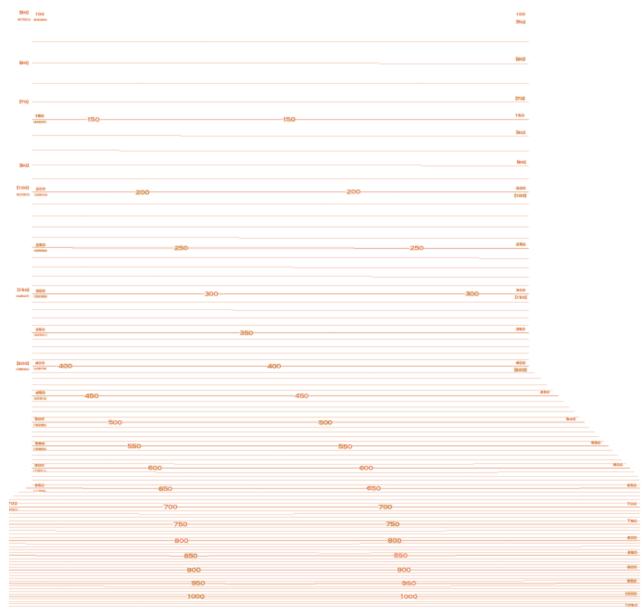
**Figure 6.** Isobaric lines in Skew-T diagram
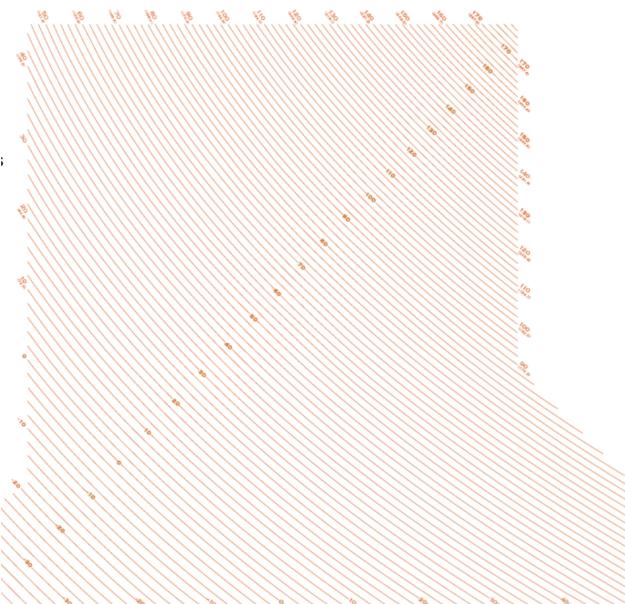
Figure 7. Dry-adiabat lines in Skew-T diagram
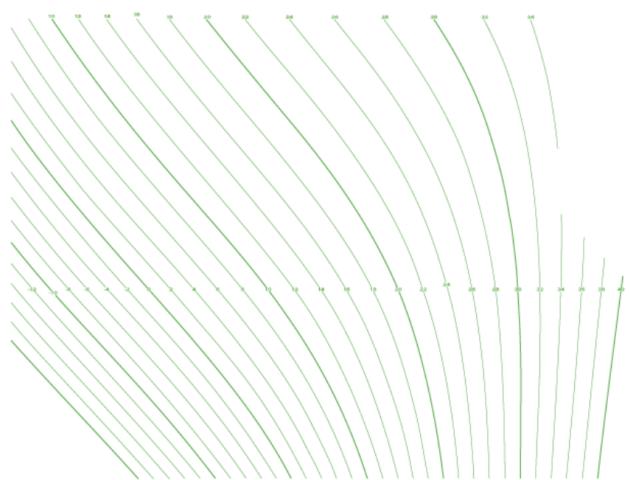

Figure 8. Moist-adiabat lines in Skew-T diagram
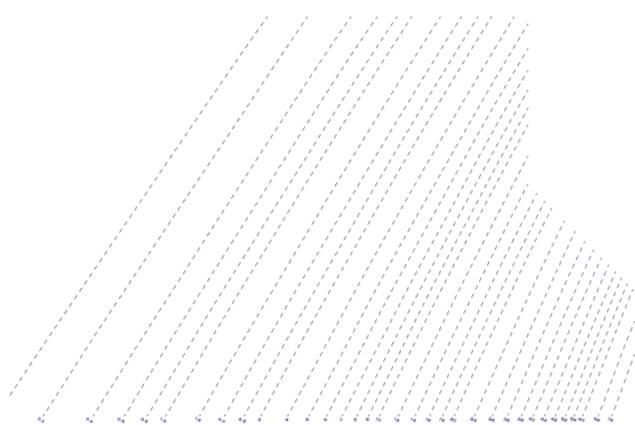
**Figure 9.** Mixing ratio lines in Skew-T diagram
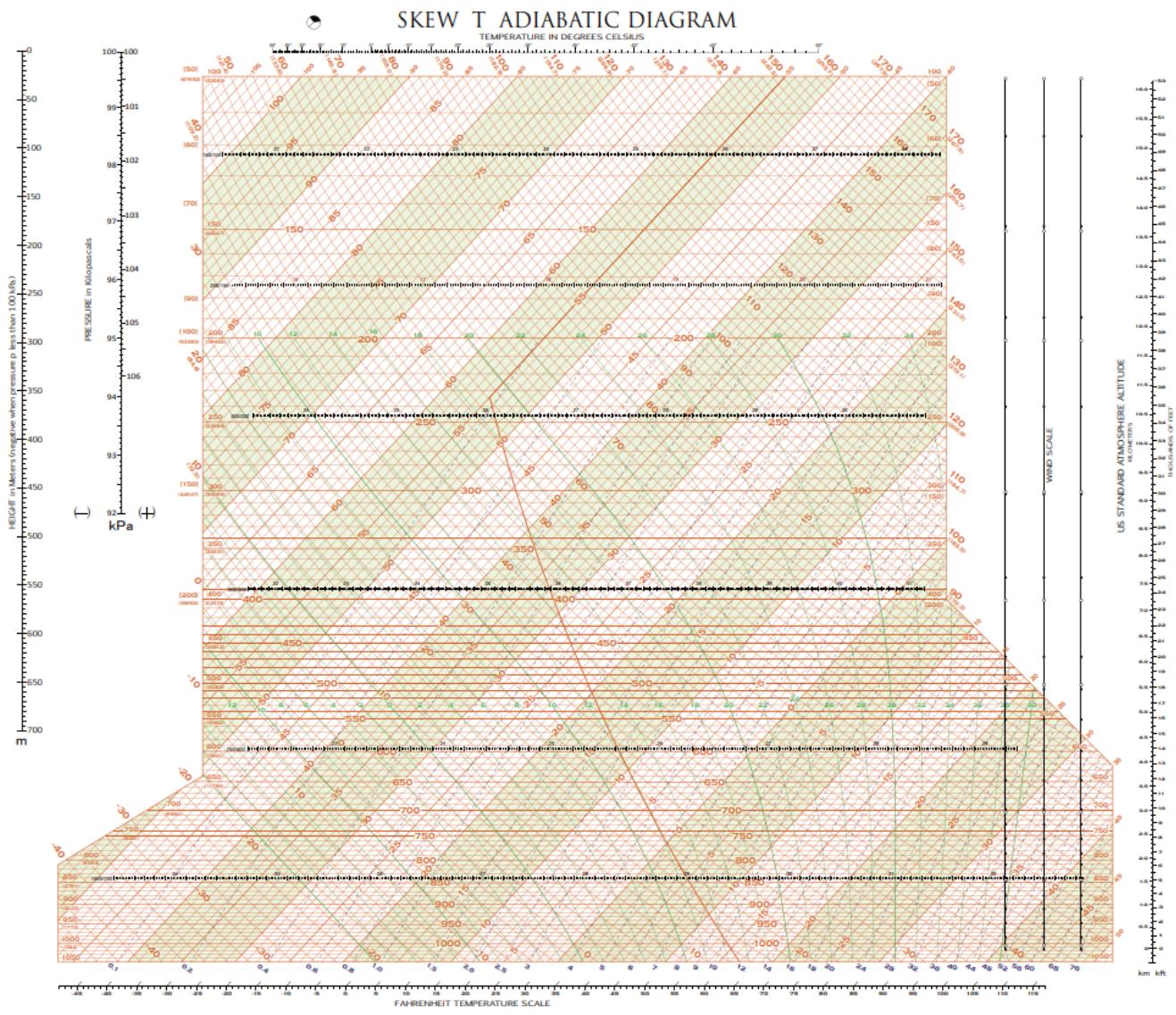
Figure 10. Skew-T diagram

15. Lecture 15 03/10/2024

15.1 . Cyclones

Properties of cyclones

1. Centre of cylce is called Eye of cyclone.
2. Cumulonimbus clouds forms the eye ball of cyclone.
3. Maximum height to where it can reaches is Tropopause
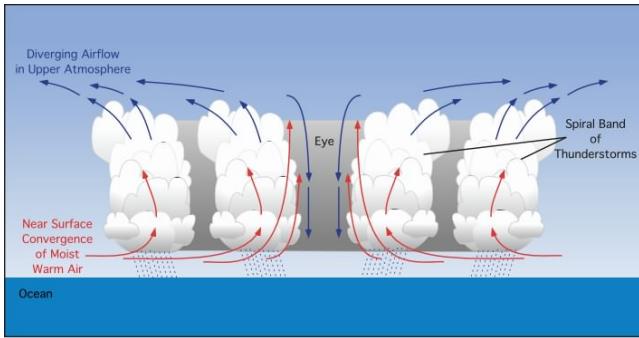
Figure 11. Internal Structure of cyclone

Image source: Internal Structure of cyclone[2]
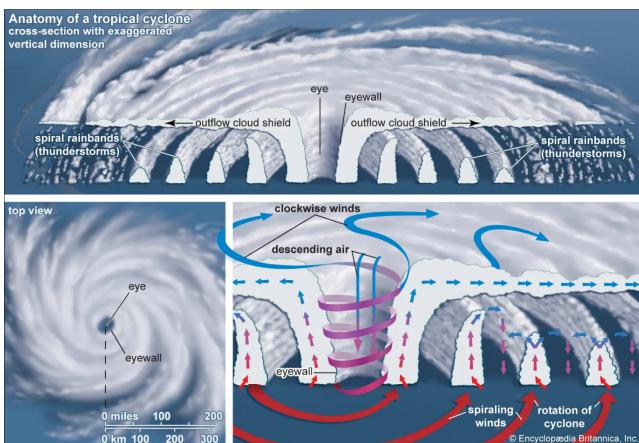
Figure 12. Air circulation in cyclone

Image source: Air circulation in cyclone[3]

15.2 . Specific enthaphy

Enthaphy is heat content of state.

$$H = U + PV \tag{313}$$
$$\Delta H = \Delta U + \Delta(PV) \tag{314}$$
$$\Delta H = \Delta U + P\Delta V + V\Delta P \tag{315}$$
$$\Delta H = Q - W + P\Delta V + V\Delta P \tag{316}$$
$$\Delta H = Q - \cancel{P\Delta V} + \cancel{P\Delta V} + V\Delta P \tag{317}$$
$$\Delta H = Q + V\Delta P \tag{318}$$

At constant pressure

$$\Delta H = Q \tag{319}$$
$$h = u + P\alpha \tag{320}$$
$$dh = du + d(PV) \tag{321}$$
$$dh = du + Pd\alpha + \alpha dP \tag{322}$$
$$dh = \delta q + \alpha dP \tag{323}$$
$$\delta q = dh - \alpha dP \text{ also,} \tag{324}$$
$$\delta q = C_p dT - \alpha dP \tag{325}$$

From Eq.(324) and Eq.(325)

$$dh - \alpha dP = C_p - \alpha dP \tag{326}$$

At constant Pressure

$$dh = C_p dT \tag{327}$$

Enthalphy is "sensible heat"

Conservative property
For a hydrostatic atmosphere:

$$\frac{dp}{dz} = \rho g \Rightarrow dp = -\rho g dz \tag{328}$$

Substitute Eq.(325) in Eq.(328)

$$\delta q = C_p dT - \cancel{\alpha}\cancel{\rho} g dz \tag{329}$$
$$\delta q = C_p dT + g dz \tag{330}$$
$$\delta q = dh + d\phi \tag{331}$$
$$\delta q = d(h + \phi) \tag{332}$$
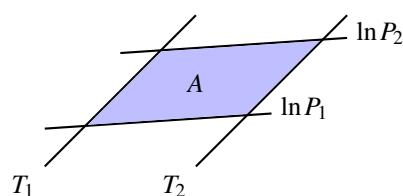
If process is adiabatic $\delta q = 0$

$$h + \phi = \text{const} \tag{333}$$

Eq.(333) called dry static equation.
This implies that when air parcel goes up (in geopotential) by the expense of heat content (enthalphy), i.e. $[h \uparrow \phi \downarrow]$.

$$\text{Area } A \propto (T_2 - T_1)(\ln P_2 - \ln P_1) \tag{334}$$

$$\propto (T_2 - T_1) \ln \left(\frac{P_2}{P_1} \right) \tag{335}$$

Cycle 1:

$$w_1 = \int P \, d\alpha$$

$$= P \int_{\alpha_1}^{\alpha_2} d\alpha$$

$$= P(\alpha_2 - \alpha_1)$$

$$= R_d (T_2 - T_1)$$

Cycle 2:

$$w_2 = \int P \, d\alpha$$

$$= \int_{\alpha_1}^{\alpha_2} \frac{R_d T}{\alpha} \, d\alpha$$

$$= R_d T_2 \ln \left(\frac{\alpha_2}{\alpha_1} \right)$$

$$= R_d T_2 \ln \left(\frac{P_1}{P_2} \right)$$

Cycle 3:

$$w_3 = \int P \, d\alpha$$

$$= P \int_{\alpha_2}^{\alpha_1} d\alpha$$

$$= P(\alpha_1 - \alpha_2)$$

$$= R_d (T_1 - T_2)$$

$$= -R_d (T_2 - T_1)$$

Cycle 4:

$$w_4 = \int P \, d\alpha$$

$$= \int_{\alpha_2}^{\alpha_1} \frac{R_d T}{\alpha} \, d\alpha$$

$$= R_d T_1 \ln \left(\frac{\alpha_1}{\alpha_2} \right)$$

$$= R_d T_1 \ln \left(\frac{P_2}{P_1} \right)$$

$$= -R_d T_1 \ln \left(\frac{P_1}{P_2} \right)$$

$$w_{\text{net}} = w_1 + w_2 + w_3 + w_4$$

$$= R_d (T_2 - T_1) + R_d T_2 \ln \left(\frac{P_1}{P_2} \right) - R_d (T_2 - T_1) - R_d T_1 \ln \left(\frac{P_1}{P_2} \right)$$

$$\Rightarrow w_{\text{net}} = 0$$

16. Lecture 16 07/10/2024

16.1 . Lesson learnt from carnot cycle

1. Thermodynamic effeciency of cyclone depends on sources, given by Eq.(312)

   When $T_1 = T_2$, carnot cycle does not exist called **Kelvin's Postulate**.

   Carnot engine is most effiecent at winter because of difference in termperature, casues diverse climate change.

2. Transformation of heat is not possible from cold body to hot body called **Clausius postulate**.

3. Carnot cycle could give us definition for absolute zero temperature.

4. All process are irreversible, however, slow process can be considered to be reversible.

16.2 . Entropy

Let's recall that,

$$q_1 = R_d T_1 \ln \left(\frac{\alpha_2}{\alpha_1} \right) \tag{336}$$

$$\frac{q_1}{T_1} = R_d \ln \left(\frac{\alpha_2}{\alpha_1} \right) \tag{337}$$

Similiarly,

$$q_2 = -R_d T_2 \ln \left(\frac{\alpha_2}{\alpha_1} \right) \tag{338}$$

$$\frac{q_2}{T_2} = -R_d \ln \left(\frac{\alpha_2}{\alpha_1} \right) \tag{339}$$

$$\frac{q_1}{T_1} + \frac{q_2}{T_2} = 0, \qquad q_1 > 0, q_2 < 0 \tag{340}$$

Effiency of heat engine:

$$\eta_{\text{rev}} = 1 - \frac{T_2}{T_1} \tag{341}$$

$$\eta_{\text{rev}} = \frac{T_1 - T_2}{T_1} \tag{342}$$

$$T_1 \cdot \eta_{\text{rev}} = T_1 - T_2 \tag{343}$$

Multiply and divide Eq.(340) by η_{rev}

$$\frac{q_1}{T_1} \cdot \frac{\eta_{\text{rev}}}{\eta_{\text{rev}}} + \frac{q_2}{T_2} = 0 \tag{344}$$

From Eq.(343),

$$q_1 \cdot \frac{\eta_{\text{rev}}}{(T_1 - T_2)} + \frac{q_2}{T_2} = 0 \tag{345}$$

The above expression is applicable only if process is **perfectly reversible**.
If process is not prefrectly reversible, i.e., irreversible

$$q_1 \cdot \frac{\eta_{\text{irrev}}}{(T_1 - T_2)} + \frac{q_2}{T_2} < 0 \tag{346}$$

In general,

$$\sum_{i=1}^{2} \frac{q_i}{T_i} \leq 0, \begin{cases} = 0 & \text{if perfectly reversible,} \\ < 0 & \text{if irreversible} \end{cases} \tag{347}$$

$$\Rightarrow \sum_{i=1}^{N} \frac{q_i}{T_i} \leq 0, \quad \text{for } N \text{ number of sources.} \tag{348}$$

If $N \to \infty$

$$\oint_{\substack{\text{cyclic} \\ \text{process}}} \frac{\delta q}{T} \leq 0 \to \text{Entropy} \tag{349}$$

$$\delta q = C_v dT + P d\alpha \tag{350}$$

$$\frac{\delta q}{T} = C_v \frac{dT}{T} + P \frac{d\alpha}{T} \tag{351}$$

$$\oint \frac{\delta q}{T} = C_v \oint \frac{dT}{T} + \oint \frac{RT}{\alpha T} d\alpha \tag{352}$$

$$ds = \oint \frac{\delta q}{T} = C_v \oint \frac{dT}{T} + R \oint d\ln\alpha \tag{353}$$

$$ds = \oint \frac{\delta q}{T} = C_v \oint d\ln T + R \oint d\ln\alpha \tag{354}$$

$$\Rightarrow ds = s_f - s_i = \oint_i^f \frac{\delta q}{T} \tag{355}$$

Number of states in which the system can have large disorder hence higher entropy.
(No. of molecule ↑) (No. of possible system states ↑) → (Entropy↑)
//

> **Question 16.1: Show that entropy of ideal gas depends on intial and final state of temperature and volume.**
>
> $$\Rightarrow ds = \oint \frac{dq}{T} = C_v \oint d(\ln T) + R \oint d(\ln \alpha)$$
>
> $$\Delta S = \int_i^f ds = S_f - S_i = C_v \ln \left(\frac{T_f}{T_i} \right) + R \ln \left(\frac{\alpha_f}{\alpha_i} \right)$$
>
> $$S_f = S_i + C_v \ln \left(\frac{T_f}{T_i} \right) + R \ln \left(\frac{\alpha_f}{\alpha_i} \right)$$
>
> $$S_f = S_i + C_v \ln \left(\frac{T_f}{T_i} \right) + C_v \left(\frac{R}{C_v} \right) \ln \left(\frac{\alpha_f}{\alpha_i} \right)$$
>
> $$S_f = S_i + C_v \left[\ln \left(\frac{T_f}{T_i} \right) \times \left(\frac{\alpha_f}{\alpha_i} \right)^{\left(\frac{C_p - C_v}{C_v} \right)} \right]$$
>
> $$S_f = S_i + C_v \left[\ln \left(\frac{T_f}{T_i} \right) \times \left(\frac{\alpha_f}{\alpha_i} \right)^{(\gamma - 1)} \right] \text{ where } \gamma = C_p / C_v$$

17. Lecture 17 10/10/2024

17.1 . Entropy

In last lecture we derived expression of entropy, i.e.
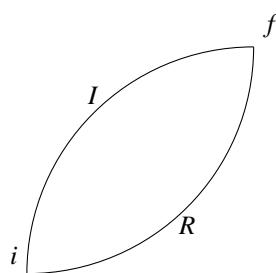
$$\oint \frac{q}{T} \leq 0$$

$\rightarrow$ Depends on the final and initial states of temperature and volume.

Let us consider a cyclic process with initial state i and final state f, the path through which process occur be denoted by R and I representing reversible and irrreversible processes.

$$i \xrightarrow{R} f \quad \text{Reversible}$$

$$i \xrightarrow{I} f \quad \text{Irreversible}$$

As we know from any cyclic process

$$\oint \frac{q}{T} \leq 0 \tag{356}$$

$$\left[\int_i^f \frac{q}{T} \right]_R + \left[\int_f^i \frac{q}{T} \right]_I \leq 0 \tag{357}$$

Since,

$$ds = \frac{\delta q}{T} \tag{358}$$

$$S_f - S_i + \left[\int_f^i \frac{q}{T} \right]_I \leq 0 \tag{359}$$

$$S_f - S_i \geq \left[\int_i^f \frac{q}{T} \right]_I \tag{360}$$

$$ds \geq \left[\int_i^f \frac{q}{T} \right]_I \tag{361}$$

$$T ds \geq \delta q \tag{362}$$

It indicates that the upperbound of the heat abosorbedby the system during a given changes.

For an isolated system,

$$\delta q = 0 \tag{363}$$

$$S_f - S_i \geq 0 \tag{364}$$

$$S_f \geq S_i \tag{365}$$

For a spontaneous irreversible transformation, occuring in an isolated system, the final entropy is greater then initial entropy.

17.2 . 2nd law of themodynamics

2nd law of themodynamics can be stated as:

1. For reversible transformation, there is no change in entropy of universe.

2. The entropy of universe increase as a result of irreversible transformation.

$$\Delta S_{\text{universe}} = \Delta S_{\text{system}} + \Delta S_{\text{surrounding}}$$

$$\Delta S_{\text{surrounding}} = 0 \quad \text{For reversible transformation}$$

$$\Delta S_{\text{surrounding}} > 0 \quad \text{For irreversible transformation}$$

//

Question 17.1: Calculate the change in air pressure if the specific entropy decrease by $0.05 J g^{-1} K^{-1}$ and the air temperature decreases by 5%.

$\Rightarrow$ To calculate the change in air pressure given the specific entropy decrease and temperature change

Given:

- Change in specific entropy: $ds = -0.05\,\mathrm{J g^{-1} K^{-1}}$
- Temperature decrease: $dT = -0.05T$ (a decrease of 5%)

Assuming the process to be reversible

Using the equation:

$$ds = C_p\, d\ln(T) - R\, d\ln(P)$$

Substituting values:

$$dT = -0.05T \Rightarrow d\ln(T) = \frac{dT}{T} = -0.05$$

$$ds = C_p(-0.05) - R\, d\ln(P)$$

$$-0.05 = 1005(-0.05) - 287\, d\ln(P)$$

$$-0.05 = -50.25 - 287\, d\ln(P)$$

$$287\, d\ln(P) = -50.25 + 0.05$$

$$287\, d\ln(P) = -50.20$$

$$d\ln(P) = \frac{-50.20}{287} \approx -0.174$$

Integrating:

$$P_f = P_i e^{-0.174}$$

where P_f is the final pressure and P_i is the initial pressure. ds $= C_p d\ln(T) - R d\ln(P)$

Recall that,

$$\theta = T\left(\frac{1000}{P}\right)^{\frac{R_d}{C_p}} \tag{366}$$

$$\ln\theta = \ln T + \frac{R_d}{C_p}\ln(1000) - \frac{R_d}{C_p}\ln P \tag{367}$$

$$C_p d(\ln\theta) = C_p d(\ln T) - R_d d(\ln P) \tag{368}$$

From 1st law of thermodynamic,

$$\delta q = C_p dT - \alpha dP \tag{369}$$

$$ds = \frac{C_p dT}{T} - \frac{\alpha dP}{T} \tag{370}$$

$$ds = \frac{C_p dT}{T} - \frac{R_d dP}{P} \tag{371}$$

$$ds = C_p d\ln(T) - R_d d\ln(P) \tag{372}$$

Comparing Eq.(368) and Eq.(368) we can find that,

$$ds = C_p d(\ln\theta) \tag{373}$$

$$S = C_p \ln\theta + \text{const.} \tag{374}$$

Therefore, lines of constant potential temperature are lines of constant entropy, but not if process becomes irreversible. Specific entropy is given by logarithm of potential temperature, when θ remains constant $\implies$ entropy remains constant. For irreversible transformation

$$ds > 0 \quad d\theta = 0$$

All isentropic process are adiabatic but all adiabatic process are not isentropic.

//

Question 17.2: During a process a parcel of dry air decent from $900hPa$ to $950hPa$ and if specific entropy decreases by $30Jkg^{-1}K^{-1}$. If it's initial temperature is $273K$; What is it's final temperature and potential temperature?

$\Rightarrow$ The relationship between specific entropy S, temperature T, and pressure P for an ideal gas assuming the process to be reversible can be expressed as:

$$ds = C_p d\ln(T) - R d\ln(P)$$

Given:

$$P_i = 900hPa,$$
$$P_f = 950hPa,$$
$$T_i = 273K,$$

Substituting values:

$$-30 = 1005\ln\left(\frac{T_f}{273}\right) - 287\ln\left(\frac{950}{900}\right)$$

$$-14.4827 = 1005\ln\left(\frac{T_f}{273}\right)$$

$$T_f = 273 \times e^{-0.0144}$$

$$T_f = 269.094K$$

Potential Temperature:

$$\theta = T_f\left(\frac{P_0}{P_f}\right)^{R/C_p}$$

Substituting the values:

$$\theta = 269.7 \times \left(\frac{950}{900}\right)^{0.2855} \approx 273.6K$$

$\therefore$ The final temperature is $269K$ and potential temperature is $273.6K$

18. Lecture 18 10/10/2024

18.1 . Special Forms of 2rd law of Thermodynamics

1. For finite isothermal transformation

$$\Delta U = 0 \tag{375}$$

$$\Delta S \geq \int \frac{\delta q}{T} \tag{376}$$

$$\Delta S \geq \frac{1}{T} \int \delta q \tag{377}$$

$$\Delta S \geq \frac{q}{T} \tag{378}$$

$$\Delta S \geq \frac{w}{T} \tag{379}$$

2. For adiabatic transformation

$$\Delta S \geq 0 \tag{380}$$

3. For Isochoric transformation

$$\Delta S \geq C_v \frac{dT}{T} \tag{381}$$

$$\Delta S \geq C_v \ln\left(\frac{T_f}{T_i}\right) \tag{382}$$

4. For Isobaric transformation

$$\Delta S \geq C_p \frac{dT}{T} \tag{383}$$

$$\Delta S \geq C_p \ln\left(\frac{T_f}{T_i}\right) \tag{384}$$

5. Combination of 1st and 2nd law

$$\delta q = C_p dT - \alpha dp \tag{385}$$

$$ds \geq \frac{q}{T} \tag{386}$$

$$T ds \geq C_p dT - \alpha dp \tag{387}$$

$$T ds \geq dh - \alpha dp \tag{388}$$

$$\tag{389}$$

similiarlly,

$$\delta q = C_v dT - \alpha dp \tag{390}$$

$$ds \geq \frac{q}{T} \tag{391}$$

$$T ds \geq C_v dT - \alpha dp \tag{392}$$

$$T ds \geq du - \alpha dp \tag{393}$$

18.2 . Moist Processes

There are three processes in moist gases:

- Saturation
- Sub-saturation
- Supersaturation

Evaporation: Some water molecules have sufficient kinetic energy to break free from the intermolecular forces of attraction.

Condensation: When water vapor cools down, molecules lose kinetic energy and form liquid droplets due to intermolecular attractions.
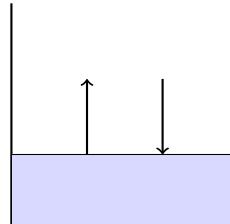


Figure 13. Saturated state
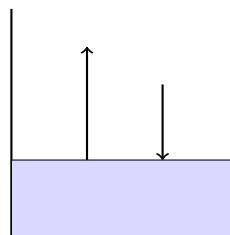

Figure 14. Sub-Saturated state
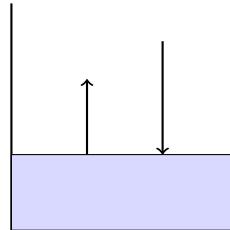
Figure 15. Super-Saturated state

For saturated state rate of evaporation is equals to rate of condensation.

Saturation vapor pressure(e_s) depends only on temperature.

Boiling Point: The temperature at which vapor pressure is equal to atmospheric temperature at the pressure of $1013 hPa$.

18.3 . Relative humidity

Relative humidity is defined as ratio of vapor pressure to saturation vapor pressure.
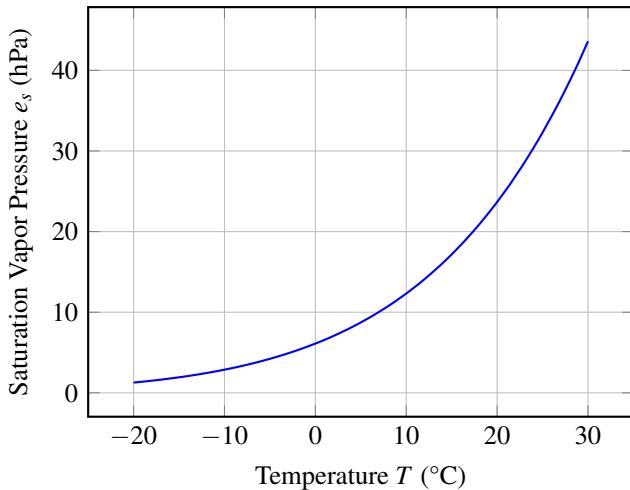
$$\text{Relative humidity (RH)} = \frac{e}{e_s} \times 100\% \tag{394}$$

$$e < e_s(T) \to \text{Sub-Saturation} \tag{395}$$

$$e = e_s(T) \to \text{Saturation} \tag{396}$$

$$e > e_s(T) \to \text{Super-Saturation} \tag{397}$$

*Note:* **Specific humidity is associated with mass**

Figure 16. Saturation Vapor Pressure e_s vs Temperature T

18.4 . Dew point temperature

There are 2 ways to make sub-saturated to saturated:

1. Reduce temperature
2. Add moisture, so that vapor pressure increase.

If moisture amount remains constant and temperature is reduced, thus saturation is achieved and this temperature is called **Dew point temperature**.

18.5 . Latent heat

Total energy required to convert unit mass from one phase to another.
Specific enthapy of phase change.
Latent heat of water at STP is $\approx 10^6 J/Kg$ (Depends on temperature)

Latent heat of evaporation

Latent heat of evaporaton of water at:

- $T = -40°C,$ Latent heat $= 2.6 \times 10^6 J/Kg$
- $T = 0°C,$ Latent heat $= 2.5 \times 10^6 J/Kg$
- $T = 40°C,$ Latent heat $= 2.26 \times 10^6 J/Kg$

Latent heat of fusion

Latent heat of fusion of water at STP $= 3.3 \times 10^6 J/Kg$

Latent heat of sublimation

Latent heat of sublimation of water at STP $= 2.83 \times 10^6 J/Kg$

Question 18.1: On a winter day the ouside air have temperature of -15°C and relative humidity of 70%.
a. If the outside air is brought inside and heated to room temperature of 20°C without adding moisture. What is new relative humidity?
b. If the room volume is 60m^3 then what mass of water must be added to the air by the humidifier to raise the relative humidity to 40%?
c. How heating is needed to accomplish a. and b.?

$\Rightarrow\Rightarrow a.$The saturation vapor pressure at -15°C can be found usinglookup tables.
For $T = 15°C$, the saturation vapor pressure,

$$e_s(-15°C) \approx 1.93 hPa$$

Given the relative humidity is 70%, the actual vapor pressure at 15°C is:

$$e = \frac{70}{100} \times 1.93 hPa \approx 1.35 hPa$$

For $T = 20°C$, the saturation vapor pressure,

$$e_s(20°C) \approx 23.37 hPa$$

Using the actual vapor pressure calculated earlier $e = 1.35$hPa and the new saturation vapor pressure $e_s(20°C)$:

$$RH\, at\, 20°C = \frac{e}{e_s} \times 100\% = \left(\frac{1.35 hPa}{23.37 hPa}\right) \times 100\% \approx 5.8\%$$

$\Rightarrow b.$To find mass of water needed to raise RH to 40%

$$e = \frac{40}{100} \times 23.37 hPa = 9.35 hPa$$

The amount of water vapor needed to raise the humidity is proportional to the difference in vapor pressure:

$$\Delta e = 9.35 hPa - 1.35 hPa = 8.00 hPa$$

The mass of water vapor can be calculated using the ideal gas law:

$$m = \frac{\Delta e V}{R_v T}$$

$$m = \frac{800 Pa \times 60 m^3}{461.5 Jkg^{-1}K^{-1} \times 293K} \approx 0.35 kg$$

$\Rightarrow c.$Heating Needed to Accomplish (a) and (b)

$$m_{\text{air}} = \frac{95000\,\text{Pa} \times 60\,\text{m}^3}{287\,\text{Jkg}^{-1}\text{K}^{-1} \times 258\,\text{K}} \approx 77.7\,\text{kg}$$

$$Q_{\text{air}} = 77.7 \times 1005 \times 35 \approx 2.73 \times 10^6 J$$

$$Q_{\text{water}} = 0.35 \times 2.5 \times 10^6 = 0.875 \times 10^6 J$$

$$Q_{\text{total}} = 2.73 \times 10^6 + 0.875 \times 10^6 \approx 3.61 \times 10^6 J$$

19. Lecture 19 21/10/2024

19.1 . The Clausius-Clapeyron equation

Let L is latent heat associated with phase change of a liquid(i) to vapor(f) state:

$$L = \int dq \tag{398}$$

$$L = \int_i^f du + \int_i^f P d\alpha \tag{399}$$

$$L = u_f - u_i + e_s(\alpha_f - \alpha_i) \tag{400}$$

where $e_s \rightarrow$ saturation vapor pressure
When phase cahnge happen, temperature remains constant, therefore

$$L = T \int \frac{\delta q}{T} \tag{401}$$

$$L = T \int_i^f ds \tag{402}$$

$$L = T(S_f - S_i) \tag{403}$$

From Eq.(400) and Eq.(403)

$$T(S_f - S_i) = U_f - U_i + e_s(\alpha_f - \alpha_i) \tag{404}$$
$$TS_f - U_f - e_s\alpha_f = TS_i - U_i - e_s\alpha_i \tag{405}$$
$$U + e_s\alpha - TS = \text{Const.} = G \tag{406}$$

where G is Gibbs energy
However, Gibbs energy is not constant when pressure and temperature changes.

$$dG = du + e_s d\alpha + \alpha de_s - T ds - s dT \tag{407}$$
$$dG = \cancel{Tds} + \alpha de_s - \cancel{Tds} - s dT \tag{408}$$
$$dG = \alpha de_s - s dT \tag{409}$$

consider a situation when liquid phase and vapor phase are in equilibrium

$$dG_1 = dG_2 \tag{410}$$
$$\alpha_i de_s - S_i dT = \alpha_f de_s - S_f dT \tag{411}$$
$$\frac{S_f - S_i}{\alpha_f - \alpha_i} = \frac{de_s}{dT} \tag{412}$$
$$\frac{L}{T(\alpha_f - \alpha_i)} = \frac{de_s}{dT} \tag{413}$$

Eq.(413) is called Clasuius-Clapeyron equation.
Let initial state(i) be liquid state and final state(f) be vapor state (evaporation)

$$\frac{e_s}{dT} = \frac{L}{T(\alpha_v - \alpha_l)} \tag{414}$$

For atmosphere $\alpha_v \ll \alpha_l$. therefore,

$$\frac{e_s}{dT} \approx \frac{L}{T\alpha_v} \tag{415}$$

Using ideal gas equation $e_s\alpha_v = R_v T$,

$$\frac{de_s}{dT} = \frac{Le_s}{R_v T^2} \tag{416}$$

Integrate form of Clausius-Clapeyron equation:

$$\int_{e_0}^{e_s} \frac{e_s}{e_s} = \int_{T_0}^{T} \frac{LdT}{R_v T^2} \tag{417}$$

$$\ln \frac{e_s}{e_0} = \frac{L}{R_v}\left[\frac{1}{T_0} - \frac{1}{T}\right] \tag{418}$$

$$e_s = e_0 \exp\left(\frac{L}{R_v}\left[\frac{1}{T_0} - \frac{1}{T}\right]\right) \tag{419}$$

Standard values $T_0 = 273$ K, $e_0 = 611$ Pa or 6.11 hPa,
$L = 2.5 \times 10^6$ J/kg, $R_v = 461$ JK^{-1}kg
$e_s(T) \approx Ae^{-B/T}$, where $A = 2.53 \times 10^{11}$Pa, $B = 5420$K.
This gives more accurate equation for temperature range from $-30°$C $\leqslant T \leqslant 35°$C

$$e_s(T) = 611.2\left[\frac{17.69T_c}{T_c + 243.5}\right] \tag{420}$$

where T_c is temperature in celcius scale
For Ice,

$$e_i(T) = e_{i0}\exp\left(\frac{L_s}{R_v}\left[\frac{1}{T_0} - \frac{1}{T}\right]\right) \tag{421}$$

where T_c is temperature in celcius scale and L_s is latent heat of sublimation.
$e_s(T) \approx A_ie^{-B_i/T}$, where $A = 3.41 \times 10^{12}$Pa, $B = 6130$K.

> **Question 19.1: For T of 265K; find the value of e_i and e_s.**
>
> $\Rightarrow e_i(T) = 611.2 \times 10^{12} \times e^{-\frac{6130}{265}}$
> $= 3060.65 Pa$
>
> $\Rightarrow e_s(T) = 2.53 \times 10^{11} \times e^{-\frac{5420}{265}}$
> $= 331.56 Pa$

$e_i < e_s \rightarrow$ at subfreezing temperatures, the environment that is saturated w.r.t liquid water is supersaturated w.r.t ice.

20. Lecture 20 23/10/2024

20.1 . Saturation mixing ratio

Mixing ratio:

$$\omega = \frac{m_v}{m_d} \tag{422}$$

$$\omega = \frac{\rho_v}{\rho_d} \tag{423}$$

$$\omega = \frac{\varepsilon e}{P - e} \tag{424}$$

$$\omega \approx \frac{\varepsilon e}{P} \tag{425}$$

where $e = \frac{R_d}{R_v} = 0.622$

Saturation mixing ratio:

$$\omega_s(T,P) \approx \frac{\varepsilon e_s(T)}{P} \quad \text{Unit: g/kg} \tag{426}$$

$$\omega_s(T,P = 622\text{hPa}) \approx \frac{0.622 e_s(T)}{622} = 0.001 \text{ kg/kg} \tag{427}$$
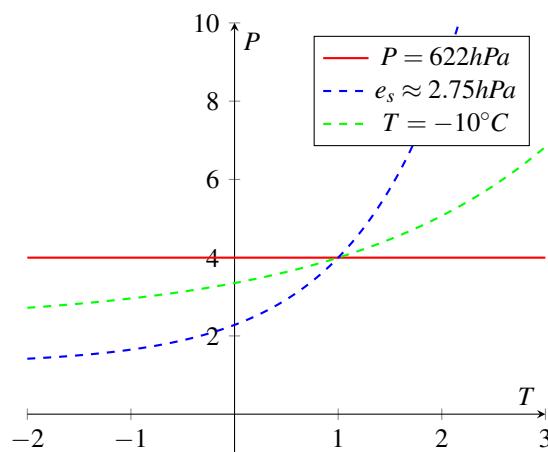
$$\omega_s(T,P = 622\text{hPa}) \approx 0.001 \text{ kg/kg} \tag{428}$$

$$\omega_s(T,P = 622\text{hPa}) \approx e_s(T) \text{ g/kg} \tag{429}$$

//

> **Question 20.1: Using skew-T diagram determine the saturation vapor pressure at -10°C.**
>
> $\Rightarrow e_s = 2.75\text{hPa}$

> **Question 20.2: Using Skew-T diagram determine the relative humidity of air parcel having temperature 20°C and dew point Temperature of 10°C at presssure 850hPa**
>
> $\Rightarrow \omega_s(20°C, 850 \text{ hPa}) = 18 \text{ hPa}$
>
> $\omega_s = \frac{\varepsilon e_s(T)}{P}$
>
> $\frac{\omega}{\omega_s} \approx \frac{e}{e_s(T)}$
>
> $RH = \frac{e}{e_s(T)} \times 100\%$
>
> $RH = \frac{\omega}{\omega_s} \times 100\%$
>
> $RH = \frac{9}{18} \times 100\%$
>
> $RH = 50\%$

//

21. Lecture 21 24/10/2024

21.1 . Lifting Conduction Level (LCL)

$$\omega_s = \frac{\varepsilon e_s(T)}{P}$$

As P decrease T decrease.
$\frac{d\omega_s}{dp}$ decides how saturation vapor pressure depends on pressure only (not on both Temperature and Pressure)

$$\frac{d\omega_s}{dP} = \frac{d}{dP}\left[\frac{\varepsilon e_s(T)}{P}\right] \tag{430}$$

$$= \varepsilon\left[\frac{1}{P}\frac{de_s(T)}{dP} - \frac{e_s}{P^2}\right] \tag{431}$$

$$= \frac{\varepsilon e_s}{P}\left[\frac{1}{e_s}\frac{de_s(T)}{dP} - \frac{1}{P}\right] \tag{432}$$

$$= \frac{\varepsilon e_s}{P}\left[\frac{1}{e_s}\frac{de_s(T)}{dT}\frac{dT}{dP} - \frac{1}{P}\right] \tag{433}$$

Using Clausius-Clapeyron equation Eq.(416)

$$\frac{d\omega_s}{dP} = \frac{\varepsilon e_s}{P}\left[\frac{1}{e_s}\frac{Le_s}{R_vT^2}\frac{dT}{dP} - \frac{1}{P}\right] \tag{434}$$

$$\frac{d\omega_s}{dP} = \frac{\varepsilon e_s}{P}\left[\frac{L}{R_vT^2}\frac{dT}{dP} - \frac{1}{P}\right] \tag{435}$$

Since process is adiabatic (Lifting of moist air):

$$C_p dT - \alpha dP = 0 \tag{436}$$

$$C_p\frac{dT}{dP} - \alpha = 0 \tag{437}$$

$$\frac{dT}{dP} = \frac{\alpha}{C_p} \tag{438}$$

$$\frac{dT}{dP} = \frac{R_dT}{C_pP} \tag{439}$$

Substiting Eq.(439) in Eq.(435), we get

$$\frac{d\omega_s}{dP} = \frac{\varepsilon e_s}{P}\left[\frac{L}{R_vT^2}\frac{\alpha}{C_p} - \frac{1}{P}\right] \tag{440}$$

$$\frac{d\omega_s}{dP} = \frac{\varepsilon e_s}{P}\left[\frac{L}{R_vT^2}\frac{R_dT}{C_pP} - \frac{1}{P}\right] \tag{441}$$

$$\frac{d\omega_s}{dP} = \frac{\varepsilon e_s}{P^2}\left[\frac{L}{R_vT}\frac{R_d}{C_p} - 1\right] \tag{442}$$
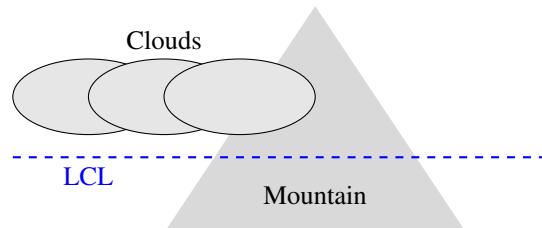
$$\frac{d\omega_s}{dP} = \frac{\omega_s}{P}\left[\frac{R_dL}{R_vC_pT} - 1\right] \tag{443}$$

$$\frac{d\omega_s}{dP} = \frac{\omega_s}{P}\left[\frac{\varepsilon L}{C_pT} - 1\right] \tag{444}$$

ω_s decreases w.r.t height (Since P decrease with height). ω_s decreases for any adiabatic uplift of air parcel.
Lifting Conduction Level is at $\omega = \omega_s$
Lifting Conduction Level is a level at which cloud start forming. It could be either Presure level or Temperature level).

Clouds

LCL

Mountain

//

> **Question 21.1:** Find out LCL for an air parcel which has a temperature of $30°C$ and dew point temperature of $0°C$ at 1000hPa.
>
> $\Rightarrow 650hPa$

//

> **Question 21.2:** Find out LCL for an air parcel which has a temperature of $25°C$ and dew point temperature of $18°C$ at 900hPa.
>
> $\Rightarrow 920hPa$

$$LCL(Km) \approx \frac{T - T_0}{8} \tag{445}$$

$$LCL(hPa) = p\exp(-0.044\Delta T_d) \tag{446}$$

where ΔT_d is dew point depression which is eequal to $(T - T_d)$. Just above LCL $\rightarrow$ phase change happen $\rightarrow$ release of latent heat which provides extra amount of energy, therefore, parcel will no longer follow dry adibat , but **moist adiabatic lapse rate**. It depends upon moisture content present in air. This process is called **pseudo-adiabatic process** because mass get lost in precipitation.

Dry adiabat lapse rate (DALR):

$$\Gamma_d = \frac{g}{c_p} = 9.8°C/km$$

Above LCL, parcel will follow **saturated adiabatic lapse rate** Γ_s which is always less than dry adiabatic lapse rate, i.e. $(\Gamma_s < \Gamma_d)$

22. Lecture 22 25/10/2024

22.1 . Saturation Adiabatic Lapse Rate

$$\delta q = C_p dT - \alpha dP \tag{447}$$

$\delta q = 0$ since process is adiabatic for dry adiabat Γ_d, but for aturation adiabatic (Γ_s) $\delta q \neq 0$
Additon of heat from latent heat of water vapor

$$\delta q = -L d\omega_s \tag{448}$$

where ω_s is saturation mixing ratio.
Therefore,

$$-L d\omega_s = C_p dT - \alpha dP \tag{449}$$

The term ω_s changes w.r.t pressure and temperature,

$$d\omega_s = \frac{\omega_s}{\partial P} dP + \frac{\omega_s}{\partial T} dT \tag{450}$$

Using the expression $\omega_s \approx \frac{\varepsilon e_s}{P}$

$$d\omega_s = -\frac{\varepsilon e_s}{P^2} dP + \frac{\varepsilon e_s}{P} \frac{1}{e_s} \frac{de_s}{dT} dT \tag{451}$$

$$d\omega_s = -\omega_s \frac{dP}{P} + \omega_s \frac{L}{R_v T^2} dT \tag{452}$$

$$-L d\omega_s = L\omega_s \frac{dP}{P} - \omega_s \frac{L^2}{R_v T^2} dT \tag{453}$$

$$c_p dT - \alpha dP = L\omega_s \frac{dP}{P} - \frac{L^2 \omega_s}{R_v T^2} dT \tag{454}$$

Invoking hydrostatic approximation

$$\frac{\partial P}{\partial z} = -\rho g \tag{455}$$

$$g dz = -\alpha dP \tag{456}$$

$$\frac{c_p dT}{g dz} - \frac{\alpha dP}{-\alpha dP} = \frac{L\omega_s \frac{dP}{P}}{-\alpha dP} - \frac{\frac{L^2 \omega_s}{R_v T^2} dT}{g dz} \tag{457}$$

$$\frac{c_p dT}{g dz} - \frac{\cancel{\alpha dP}}{-\cancel{\alpha dP}} = \frac{L\omega_s \frac{\cancel{dP}}{P}}{-\cancel{\alpha dP}} - \frac{\frac{L^2 \omega_s}{R_v T^2} dT}{g dz} \tag{458}$$

$$\frac{c_p dT}{g dz} + 1 = \frac{L\omega_s}{-\alpha P} - \frac{L^2 \omega_s dT}{R_v T^2 g dz} \tag{459}$$

$$\left(\frac{L^2 \omega_s}{R_v g T^2} + \frac{c_p}{g} \right) \frac{dT}{dz} = \frac{L\omega_s}{\alpha P} + 1 \tag{460}$$

Recall $\Gamma_d = \frac{g}{c_p} \frac{T'}{T} \approx \frac{g}{c_p}$
where T' is temperature of parcel and T is temperature of atmosphere.
Multipling $\frac{g}{c_p}$ to Eq.(460)

$$\left(\frac{L^2 \omega_s}{R_v g T^2} \frac{g}{c_p} + \frac{c_p}{g} \frac{g}{c_p} \right) \frac{dT}{dz} = \frac{L\omega_s}{\alpha P} \frac{g}{c_p} + \frac{g}{c_p} \tag{461}$$

Using ideal gas eq, $P\alpha = R_d T$

$$\left(\frac{L^2 \omega_s}{R_v c_p T^2} + 1 \right) \frac{dT}{dz} = \frac{L\omega_s}{R_d T} \Gamma_d + \Gamma_d \tag{462}$$

$$\Gamma_s = \frac{dT}{dz} = \frac{\Gamma_d \left(\frac{L\omega_s}{R_d T} + 1 \right)}{\left(\frac{L^2 \omega_s}{R_v c_p T^2} + 1 \right)} \tag{463}$$

If $\omega_s = 0$, which mean air is not moist$\rightarrow \Gamma_s = \Gamma_d = 9.8°C/km$

$$\Gamma_s = \frac{d\ln T}{d\ln P} \approx \frac{\left(\frac{L\omega_s}{R_d T} + 1 \right)}{\left(\frac{L^2 \omega_s}{R_v c_p T^2} + 1 \right)} \frac{R_d}{c_p} \tag{464}$$

$\Gamma_s \approx 6.5°C/km$ which can vary between 4 to 7 depending on amount of moisture content in air.
Note: Potential temperature is no more conserved quantity above LCL.

22.2 . Equivalent potential temperature (θ_e)
θ_e is conserved during moist adiabatic process.

$$\theta_e = \theta \exp\left(\frac{L\omega}{c_p T_{\text{LCL}}} \right) \tag{465}$$

//

> **Question 22.1:** Using skew-T estimate θ_s of air parcel which has $30°C$ temperature and $10°C$ dew point temperature at 900hPa; which then raises adiabatically.

$\Rightarrow$ 770hPa and 650hPa

23. Lecture 23 30/10/2024

### 23.1 . Psychrometer
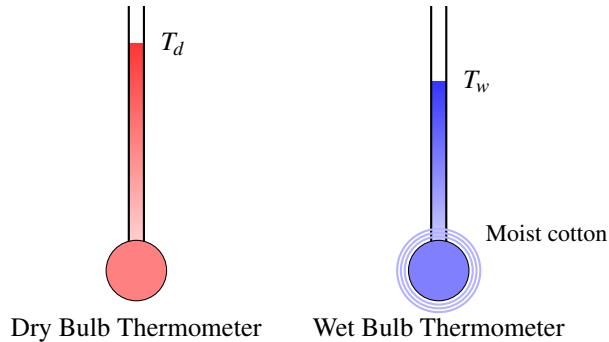
Figure 17. Psychrometer

At saturated air (100% moist air) $\rightarrow T_d = T_w$
$\therefore$ It gives information about relative humidity.
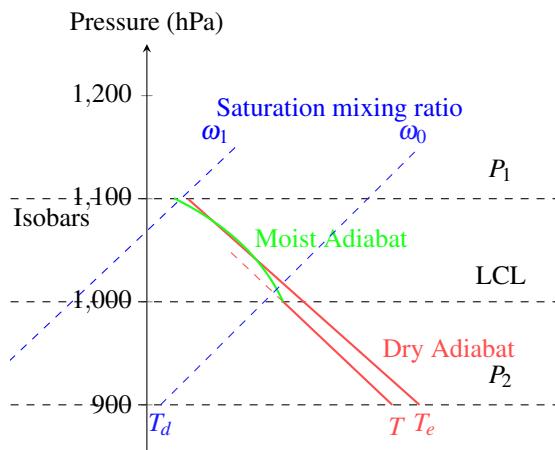
Figure 18. Skew-T diagram with Isobars, Dry and Moist Adiabat

23.2 . Normand's rule

$$T_d \leq T_w \leq T \tag{466}$$

where

T_d is dew point temperature.

T_w is wet bulb temperature.

T is parcel temperature.

24. Lecture 24 04/11/2024

24.1 . Adiabatic cloud water content

Water vapor mixing ratio:

$$\omega = \frac{m_v}{m_d} \tag{467}$$

Liquid water mixing ratio:

$$\omega_l = \frac{m_l}{m_d} \tag{468}$$

Total mixing ratio:

$$\omega_{\text{Total}} = \omega + \omega_l \tag{469}$$

Assumption considered are:

- Air contains only vapor and liquid water. There is no ice in air.
- Water does not precipitate during the process.
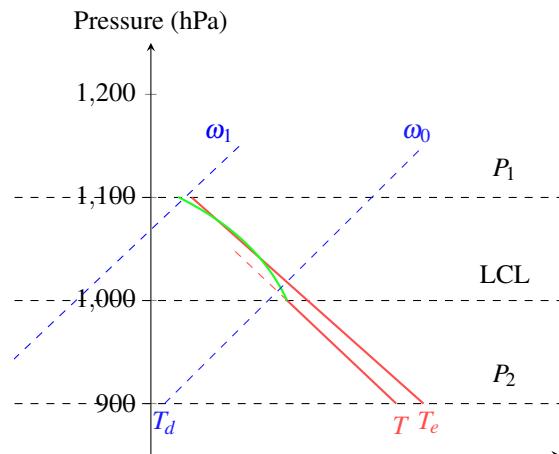
Figure 19. Skew-T diagram with Isobars, Dry and Moist Adiabat
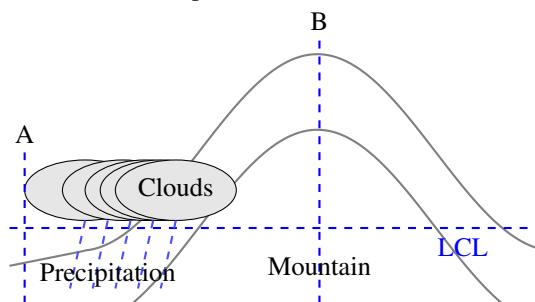
Liquid water content (LWC)

$$\text{LWC} = (\omega_0 - \omega_1) \tag{470}$$

24.2 . Flow over terrain

1. Moist adaibatic process

2. Pseudo adaibatic process

//

> **Question 24.1:** Consider a situation in which moist air from Arabian sea is advented to western ghats undergoing irreversible pseudo-adiabatic process and the pressure level at which parcel asent is 850hPa and temperature of parcel is 20.6°C; dew point temperature 15.7°C. The minimum pressure at summit is 610hPa.
>
> a. Find out the initial mixing ratio and relative humidity (RH) od air parcel before the asent.
>
> b. Find out the final temperature and dew point temperature experienced by the parcel at the leeward side of the mountain at the same level.
>
> ⇒
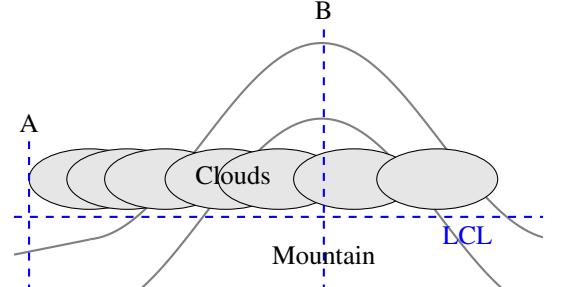
24.3 . Atmospheric stability

Hydrostatic inbalance in small-scale scenarios such as tornados.

- Stable
- Unstable
- Neutral
- Metastable

25. Lecture 25 06/11/2024

25.1 . Atmospheric stability
Over large scale, atmoshere is in hydrostatic balance

$$\frac{dp}{dz} = -\rho g \tag{471}$$
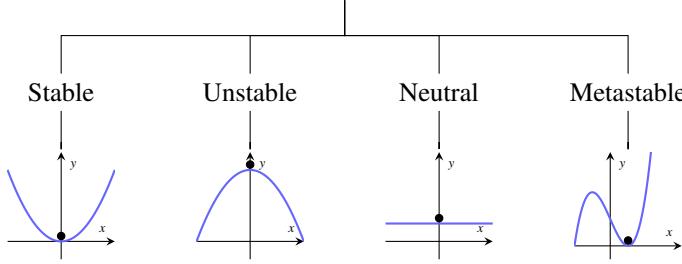
Convection/small scale process
Following assumptions are made in order to asscess the stability condition:

1. The environment is in hydrostaic equlibrium.
2. The parcel does not mix with surrounding.
3. The parcel movement does not distrub the surrounding.
4. The process is adiabatic.
5. At any given level, pressure inside the parcel equals to that of envirnment.

Three conditions

Equilibrium States

Stable   Unstable   Neutral   Metastable

25.2 . Equation of motion for the parcel
Because the environment is in hydrostatic equlibrium given by Eq.(471) This does not hold wave equation for a parcel, i.e.,

$$\rho' \frac{d^2 z}{dt^2} = -\frac{dp'}{dz} - \rho' g \tag{472}$$

$$\frac{d^2 z}{dt^2} = -\frac{1}{\rho'}\frac{dp'}{dz} - g \tag{473}$$

$$\frac{d^2 z}{dt^2} = -\alpha'\frac{dp'}{dz} - g \tag{474}$$

Using the assumption (5):

$$\frac{dp'}{dz} = \frac{dp}{dz} \tag{475}$$

$$\frac{d^2 z}{dt^2} = -\alpha'\left(\frac{-g}{\alpha}\right) - g \tag{476}$$

$$\frac{d^2 z}{dt^2} = \left(\frac{\alpha}{\alpha}\right)g - g \tag{477}$$

$$\frac{d^2 z}{dt^2} = \left(\frac{\alpha' - \alpha}{\alpha}\right)g \tag{478}$$

$$\frac{d^2 z}{dt^2} = \left(\frac{\rho - \rho'}{\rho'}\right)g \tag{479}$$

Using Ideal gas equation,

$$\frac{d^2 z}{dt^2} = \left(\frac{T_v' - T_v}{T_v}\right)g \tag{480}$$

We consider a displacement of a parcel from its mean position ,such that $z \ll 1$.
If for simplicity we take the level to be at $z = 0$, where temperature is T_{v_0}. Express T_v and T_{v_0} in terms of taylor series:

$$T_v = T_{v_0} + \frac{dT_v}{dz}\frac{z}{1!} + \frac{d^2 T_v}{dz^2}\frac{z^2}{2!} + \cdots \tag{481}$$

$$T_v' = T_{v_0} + \frac{dT_v'}{dz}\frac{z}{1!} + \frac{d^2 T_v'}{dz^2}\frac{z^2}{2!} + \cdots \tag{482}$$

Neglect the higher order terms;

$$T_v = T_{v_0} + \frac{dT_v}{dz}z \tag{483}$$

$$T_v' = T_{v_0} + \frac{dT_v'}{dz}z \tag{484}$$

$$\Gamma_v = -\frac{dT_v}{dz} \quad \rightarrow \quad \text{Environment lapse rate} \tag{485}$$

$$\Gamma_v' = -\frac{dT_v'}{dz} \quad \rightarrow \quad \text{Parcel lapse rate} \tag{486}$$

$$\frac{d^2 z}{dt^2} = g\frac{(\Gamma_v - \Gamma_v')}{(T_{v_0} - \Gamma_v z)}z \tag{487}$$

Taking $\frac{1}{T_{v_0} - \Gamma_v z}$ for simplification;

$$\frac{1}{T_{v_0} - \Gamma_v z} = \frac{1}{T_{v_0}}\left(\frac{1}{1 - \frac{\Gamma_v z}{T_{v_0}}}\right) \tag{488}$$

$$\frac{1}{T_{v_0} - \Gamma_v z} = \frac{1}{T_{v_0}}\left(1 - \frac{\Gamma_v z}{T_{v_0}}\right)^{-1} \tag{489}$$

$$\tag{490}$$

since, $\frac{\Gamma_v z}{T_{v_0}} \ll 1$, we can rewrite above equation as:

$$\frac{1}{T_{v_0} - \Gamma_v z} \approx \frac{1}{T_{v_0}}\left(1 + \frac{\Gamma_v z}{T_{v_0}}\right) \tag{491}$$

Therefore, Eq.(487)

$$\frac{d^2 z}{dt^2} \approx g(\Gamma_v - \Gamma_v') \times \frac{1}{T_{v_0}}\left(1 + \frac{\Gamma_v z}{T_{v_0}}\right) \tag{492}$$

$$\frac{d^2 z}{dt^2} \approx \frac{g}{T_{v_0}}\left(\Gamma_v z + \frac{\Gamma_v^2 z^2}{T_{v_0}} - \Gamma_v' z - \frac{\Gamma_v' \Gamma_v z^2}{T_{v_0}}\right) \tag{493}$$

Ignoring z^2 terms, we get;

$$\frac{d^2 z}{dt^2} \approx \frac{g}{T_{v_0}}(\Gamma_v - \Gamma_v')z \tag{494}$$

$$\frac{d^2 z}{dt^2} - \frac{g}{T_{v_0}}(\Gamma_v - \Gamma_v')z = 0 \tag{495}$$

The above Eq.(495) is 2nd order diffreential equation, therefore we can rewrite equation as:

$$\frac{d^2z}{dt^2} - \omega^2 z = 0 \qquad (496)$$

where $\omega = \sqrt{\frac{g}{T_{v_0}}(\Gamma_v - \Gamma'_v)}$

26. Lecture 26 07/11/2024

26.1 . Stability analysis

3 possible solutions exists:

1. Stable solution i.e. $(\Gamma'_v - \Gamma_v) > 0$

$$\frac{d^2 z}{dt^2} + \omega^2 z = 0 \qquad (497)$$

General solution is :

$$z(t) = A \sin \omega t + B \cos \omega t \qquad (498)$$

Boundary conditions : $t = 0, z = 0 \Rightarrow B = 0$

$$z(t) = A \sin \omega t \qquad (499)$$

Time period of oscillation $\Rightarrow \tau = \frac{2\pi}{\omega}$

2. Unstable solution i.e. $(\Gamma'_v - \Gamma_v) < 0$

$$\frac{d^2 z}{dt^2} - \omega^2 z = 0 \qquad (500)$$

General solution is :

$$z(t) = A \exp(\omega t) + B \exp(-\omega t) \qquad (501)$$

Boundary conditions : $t \to \infty \Rightarrow A = B \neq 0$

$$z(t) = A \exp(\omega t) \qquad (502)$$

Time period of oscillation $\to \tau = \frac{2\pi}{\omega}$

3. Neutral solution i.e. $(\Gamma'_v - \Gamma_v) = 0$

$$\frac{d^2 z}{dt} = 0 \qquad (503)$$

General solution is:

$$z(t) = At + B \qquad (504)$$

Boundary conditions : $t = 0 \Rightarrow A = 0$

$$\frac{dz(t)}{dt} = A \qquad (505)$$

Time period of oscillation is **not defined**.

For unstaurated air parcel :

1. $\Gamma_{EL} > \Gamma_d \to$ atmosphere becomes unstable.
2. $\Gamma_{EL} < \Gamma_d \to$ atmosphere becomes stable.

For staurated air parcel :

1. $\Gamma_{EL} > \Gamma_s \to$ atmosphere becomes unstable.
2. $\Gamma_{EL} < \Gamma_s \to$ atmosphere becomes stable.
3. $\Gamma_{EL} = \Gamma_s \to$ atmosphere becomes neutral.

where,

Γ_{EL} = environment lapse rate

$\Gamma_d = 9.8°C/km$

$\Gamma_s \approx 6.5°C/km$ (depends on place and humidity)

1. $\Gamma_{EL} > \Gamma_d \to$ layer is absolutely unstable.
2. $\Gamma_{EL} < \Gamma_s \to$ layer is absolutely stable.
3. $\Gamma_d < \Gamma_{EL} < \Gamma_d \to$ layer is conditionally unstable (conditionlly instablilty).
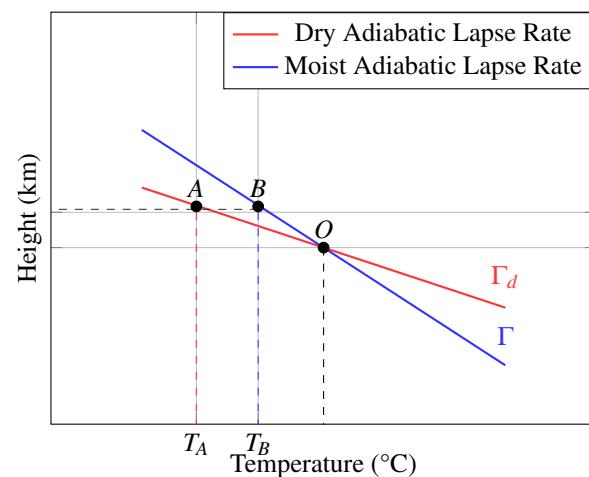

Figure 20. Case 1: $\Gamma_B > \Gamma_{dA}$
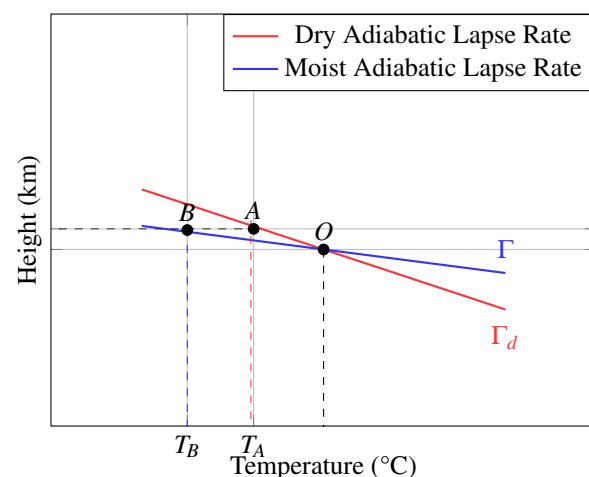

**Figure 21.** Case 2: $\Gamma_B < \Gamma_{dA}$
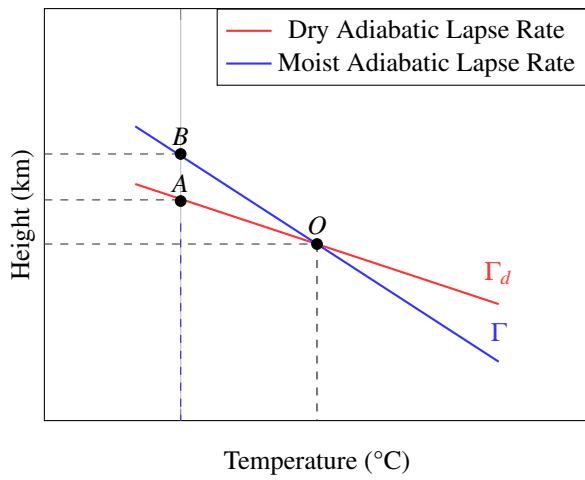
Figure 22. Case 3: $\Gamma_B = \Gamma_{dA}$

26.2 . Convective instability or Potential instability

Potential temperature (θ_e):

$$\theta_e = T \left(\frac{1000}{P} \right)^{\frac{R_d}{C_p}} \tag{506}$$

Taking logorithm on both sides,

$$\ln \theta_e = \ln T + \frac{R_d}{C_p} \ln (1000) - \frac{R_d}{C_p} \ln P \tag{507}$$

Differentiating above Equation w.r.t z

$$\frac{1}{\theta_e} \frac{\theta}{dz} = \frac{1}{T} \frac{dT}{dz} - \frac{R_d}{C_p} \left(\frac{-dp}{dz} \right) \tag{508}$$
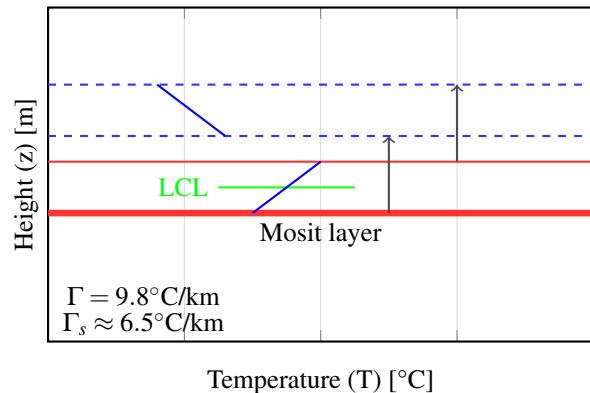
$$\frac{1}{\theta_e} \frac{\theta}{dz} = \frac{1}{T} \frac{dT}{dz} - \frac{\cancel{R_d}}{C_p \cancel{p}} \left(\frac{-\cancel{p}g}{\cancel{R_d}T} \right) \tag{509}$$

$$\frac{1}{\theta_e} \frac{\theta}{dz} = \frac{1}{T} \frac{dT}{dz} + \frac{1}{T} \left(\frac{g}{C_p} \right) \tag{510}$$
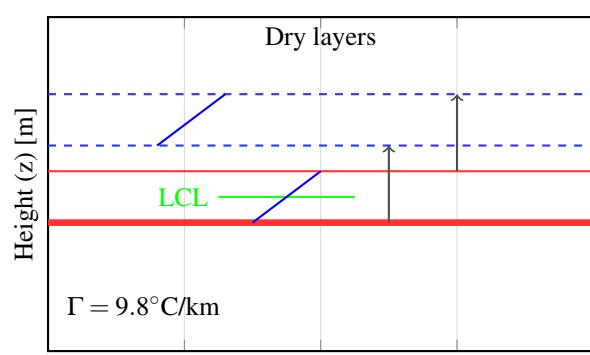
$$\frac{1}{\theta_e} \frac{\theta}{dz} = -\frac{1}{T} \Gamma + \frac{1}{T} \Gamma_d \tag{511}$$

$$\frac{1}{\theta_e} \frac{\theta}{dz} = \frac{1}{T} (\Gamma_d - \Gamma) \tag{512}$$

1. $\frac{d\theta_e}{dz} > 0 \rightarrow$ Potentially Stable
2. $\frac{d\theta_e}{dz} < 0 \rightarrow$ Potentially Unstable
3. $\frac{d\theta_e}{dz} = 0 \rightarrow$ Potentially Neutral

27. Lecture 27 08/11/2024

27.1 . Convective Available Potential Energy (CAPE) and Convective Inhibition (CIN)

$$\delta \omega = F \cdot dz \tag{513}$$

$$\delta \omega = madz \tag{514}$$

$$\delta \omega = m\frac{d^2z}{dt^2}dz \tag{515}$$

Integrating above equation to get total work done by the parcel and using Eq.(480);

$$W = \int_i^f mg\left(\frac{T_v' - T_v}{T_v}\right)dz \tag{516}$$

$$w = \int_i^f g\left(\frac{T_v' - T_v}{T_v}\right)dz \tag{517}$$

Using hydrosattic equation/approximation Eq.(94);

$$w = \int_i^f \frac{-1}{\rho}\left(\frac{T_v' - T_v}{T_v}\right)dp \tag{518}$$

Using Ideal equation ($P = \rho R_d T_v$);

$$w = \int_i^f -\left(\frac{R_d \cancel{T_v}}{P}\right)\left(\frac{T_v' - T_v}{\cancel{T_v}}\right)dp \tag{519}$$

$$w = -R_d \int_i^f (T_v' - T_v)d(\ln(P)) \tag{520}$$

Therefore,

$$\text{CAPE} = -R_d \int_{Z_{\text{LFC}}}^{Z_{\text{EL}}} (T_v' - T_v)d(\ln(P)) \tag{521}$$

$$\text{CIN} = -R_d \int_{Z=0}^{Z_{\text{LFC}}} (T_v' - T_v)d(\ln(P)) \tag{522}$$

Theritical maximum velocity

$$\frac{1}{2}w_{\text{max}}^2 = \text{CAPE} \tag{523}$$

$$w_{\text{max}} = \sqrt{2 \times \text{CAPE}} \tag{524}$$

- $< 0 \rightarrow$ stable
- $0 - 1000 \rightarrow$ marginally unstable
- $1000 - 2500 \rightarrow$ morderately unstable
- $2500 - 3500 \rightarrow$ highly unstable
- $> 3500 \rightarrow$ extremely unstable (wind speed may reach upto 50-70m/s indicating storms)

//

> **Question 27.1:** Given an air parcel at 850hPa with temperature T is $13°$C and dew point temperature T_d is $8°$C

Pressue level	Env. Temeperature
800	Data
750	Data
700	Data
650	Data
600	Data
550	Data
500	Data
450	Data
400	Data
350	Data
300	Data
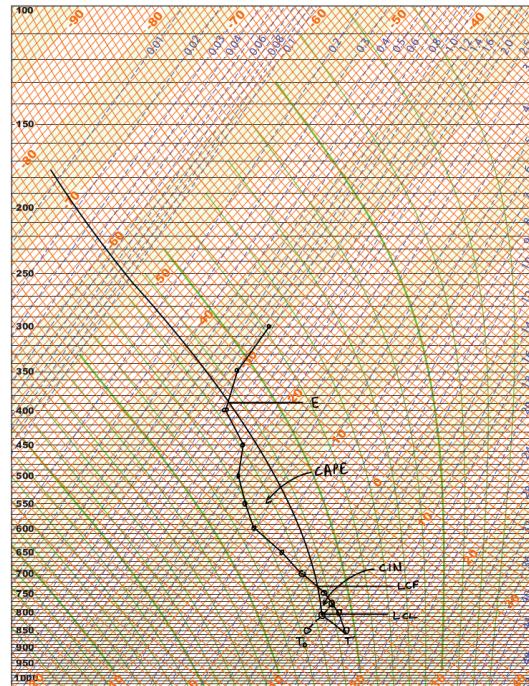
**Figure 23.** Solution of Question 27.1

28. Lecture 28 11/11/2024

- Cloud droplet size $\approx 1\mu$m.
- Rain drop size ≈ 1mm.

28.1 . Clasisus-Clapeyron Equation
Vapor to Water, condensation:
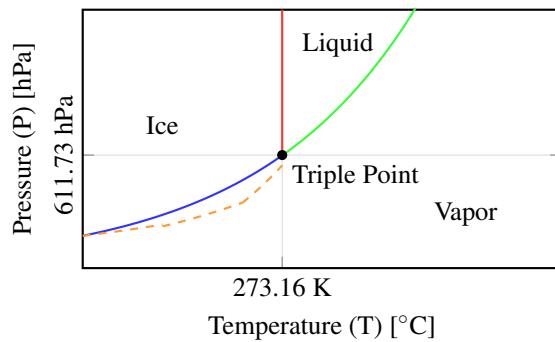
$$\frac{e_s}{T} = \frac{L}{T(\alpha_v - \alpha_w)} \approx \frac{L}{T\alpha_v} \tag{525}$$

Vapor to Ice, sublimation:

$$\frac{e_{si}}{T} = \frac{L_s}{T(\alpha_v - \alpha_i)} \approx \frac{L_s}{T\alpha_v} \tag{526}$$

Vapor to Ice, fusion:

$$\frac{p}{T} = \frac{L_f}{T(\alpha_w - \alpha_i)} \tag{527}$$


28.2 . Weigner-Bergeron-Findeisn Therory
Theory describes a process in mixed-phase clouds where ice crystals grow at the expense of surrounding supercooled water droplets. This theory explains how precipitation forms in cold clouds (those with temperatures below freezing) through the following mechanism:
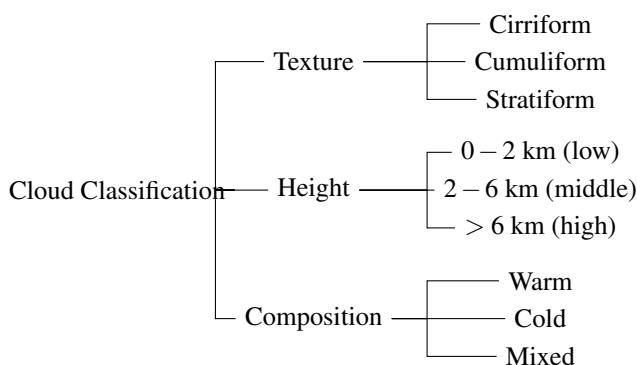
1. Different Saturation Vapor Pressures: In a cold cloud with both ice crystals and supercooled water droplets, the saturation vapor pressure over ice is lower than that over liquid water. This means that water vapor is more likely to condense onto ice than onto liquid droplets.

2. Growth of Ice Crystals: Water vapor moves from the liquid droplets (which gradually evaporate) to the ice crystals, causing the ice crystals to grow. This transfer of water vapor from droplets to ice crystals results in the gradual disappearance of liquid droplets and an increase in ice crystal size.

3. Initiation of Precipitation: As ice crystals grow, they eventually become large enough to fall from the cloud as snowflakes or, if they melt on the way down, as raindrops.

This process is particularly important in middle and high latitudes, where clouds are often mixed-phase, and it provides a primary mechanism for precipitation in cold clouds.

29. Lecture 29 13/11/2024

29.1 . Classifiaction of cloulds

Cloud Classification
- Texture
  - Cirriform
  - Cumuliform
  - Stratiform
- Height
  - $0 - 2$ km (low)
  - $2 - 6$ km (middle)
  - > 6 km (high)
- Composition
  - Warm
  - Cold
  - Mixed

29.2 . Types of particles within the clouds

Collective condensed particle within the cloud **'hydrome-teous'**

1. Cloud droplets $\to \sim 400 \, \mu$m to 10μm.
2. Raindrops $\to$ A few hundred of μm to 3nm ~ 1nm.
3. Ice cystrals $\to$ Tens of μm to hundred of μm.
4. Snow flakes $\to$ Few cm, crystalline ice particles.
5. Grapel $\to$ Few cm, when supercooled water frezzes on the surface of ice crystals.
6. Hail $\to$ Ice, around 5mm ~ 15cm.
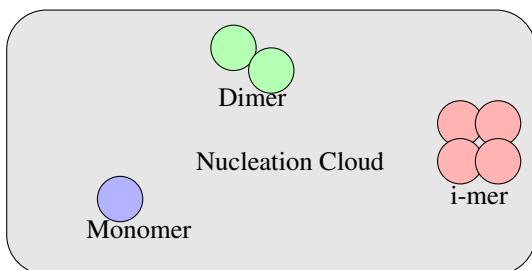
29.3 . Nucleation

Fundamental process of phase change is called as Nucleation. Nulceation are of 2 types:

1. **Homgeneous** : Involve forign particles (aerosols), CNN (Cloud Condensation Nuclei)
2. **Hetrogeneous**

Small thermal purturbation/attenuation can disintegrate i-mer. Embreyo $\to$ collection of water molecule.
Germ $\to$ activated embreyo (Started to show water droplet properties).

Dimer
Nucleation Cloud
i-mer
Monomer

One basic assumption of classic nulceation theory is that the population of i-mer embreyos follows **Boltman's distribu-**tion.

$$N_i = N \exp\left(\frac{-\Delta E_i}{KT}\right) \tag{528}$$

$$N_i = N \exp\left(\frac{\mu_v - \mu_l}{KT}\right) \tag{529}$$

$$e = e_{\text{sat}} \exp\left(\frac{\mu_v - \mu_l}{KT}\right) \tag{530}$$

$$\mu_v - \mu_l = KT \ln\left(\frac{e}{e_{\text{sat}}}\right) \tag{531}$$

where,

$$N = \text{ number of molecules}$$
$$\mu_v \text{ and } \mu_l = \text{ chemical potentials of vapor and liquid resp.}$$

Let us suppose that a small embreyoic water droplet with volume (V) and surface (A) from from a pure supersaturated water vapor.
Decrease in energy

$$\Delta E = nV(\mu_v - \mu_l) \tag{532}$$

where,

$$n = \text{No. of water vapor molecule.}$$

Work is needed to be done to create surface area:

$$A\sigma \tag{533}$$

where,

$$A = \text{ Area}$$
$$\sigma = \text{ surface energy}$$

Net increase in the energy of the system:

$$\Delta E = A\sigma - nV(\mu_v - \mu_l) \tag{534}$$
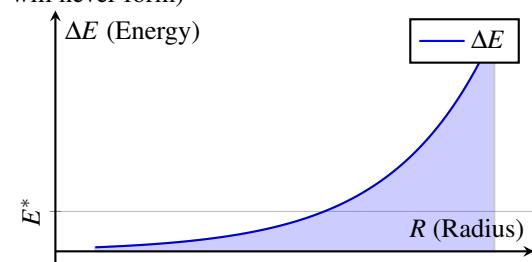
$$\Delta E = A\sigma - nVKT \ln\left(\frac{e}{e_s}\right) \tag{535}$$

If the droplet i sperical shaped with radius R.
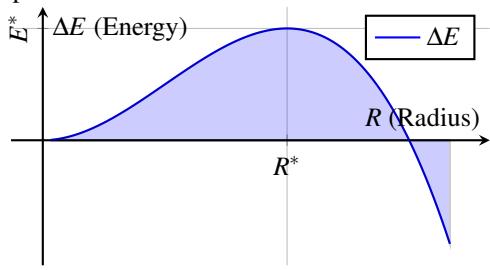
$$\Delta E = (4\pi R^2)\sigma - \frac{4}{3}\pi R^3 nKT \ln\left(\frac{e}{e_s}\right) \tag{536}$$

Where, R^* is critical radius.

1. Under subsaturated condition ($e < e_s$) $\to \Delta E > 0$ (Droplets will never form)

2. Under supersaturated condition $(e > e_s) \rightarrow \Delta E > 0$ uptill R^* then $\Delta E < 0$
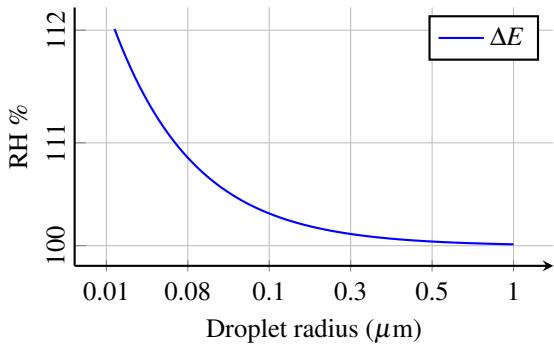
To find critical radius of embreyo (R^*):

$$\frac{d(\Delta E)}{dR} = \frac{d\left[(4\pi R^2)\sigma - \frac{4}{3}\pi R^3 nKT \ln\left(\frac{e}{e_s}\right)\right]}{dt} = 0$$

$$\text{(537)}$$

$$\frac{d(\Delta E)}{dR} = 8\pi R\sigma - 4\pi R^2 nKT \ln\left(\frac{e}{e_s}\right) = 0 \qquad \text{(538)}$$

$$R^* = \frac{2\sigma}{nKT \ln\left(\frac{e}{e_s}\right)} \qquad \text{(539)}$$

Eq.(539) is called **Kelvin's equation**

//

Question 29.2: Determine the fractional changes in (ΔE^*) and (R^*) if surface tension (σ) is reduced by 10% by adding soap in pure water.

$\Rightarrow \sigma$ decreases by 10% by adding soap to pure water

$\frac{d(\Delta E^*)}{d\sigma} = \frac{16\pi\sigma^2}{nKT \ln\left(\frac{e}{e_s}\right)} = 3 \times 0.1 = 30\%$

$\frac{d(\Delta E^*)}{\Delta E^*} = \frac{3d\sigma}{\sigma}$

$\frac{d(\Delta R^*)}{d\sigma} = \frac{2}{nKT \ln\left(\frac{e}{e_s}\right)}$
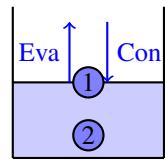
$\frac{d(\Delta R^*)}{R^*} = \frac{d\sigma}{\sigma} = 1 \times 0.1 = 10\%$

Fractional change in ΔE^* is 30% and for R^* is 10%

Question 29.1: Show that height of critical barrior (ΔE^*) is given by $\dfrac{16\pi\sigma^3}{3\left(nKT \ln\left(\frac{e}{e_s}\right)\right)^2}$.

$\Rightarrow$ At $R = R^*$

$\Delta E = 4\pi(R^*)^2\sigma - \frac{4}{3}\pi(R^*)^3 nKT \ln\left(\frac{e}{e_s}\right)$

$\Delta E = 4\pi\left(\frac{2\sigma}{nKT \ln \frac{e}{e_s}}\right)^2\sigma - \frac{4}{3}\pi\left(\frac{2\sigma}{nKT \ln \frac{e}{e_s}}\right)^3 nKT \ln\left(\frac{e}{e_s}\right)$

$\Delta E = 4\pi\frac{4\sigma^3}{\left(nKT \ln \frac{e}{e_s}\right)^2} - \frac{4}{3}\pi\frac{8\sigma^3}{\left(nKT \ln \frac{e}{e_s}\right)^2}$

$\Delta E = 4\pi\frac{12\sigma^3}{3\left(nKT \ln \frac{e}{e_s}\right)^2} - 4\pi\frac{8\sigma^3}{3\left(nKT \ln \frac{e}{e_s}\right)^2}$

$\Delta E = 4\pi\frac{4\sigma^3}{3\left(nKT \ln \frac{e}{e_s}\right)^2}$

$\Delta E = \frac{16\pi\sigma^3}{3\left(nKT \ln \frac{e}{e_s}\right)^2}$

//
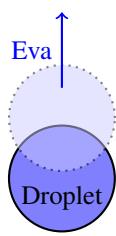
30. Lecture 30 14/11/2024
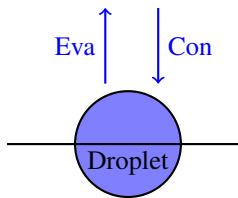
30.1 . Supersaturation with respect to plane surface of pure water

1. Larger chance of escape→rate of evaporation increases.

2. Lower chance of escape→rate of evaporation decreases.

$\therefore$ Saturation vapor pressure of pure water will be different for different geometry.

The vapor pressure adjacent to solution droplet is reduced as compared to pure water droplet.

30.2 . Köhler Equation

The Köhler equation describes the equilibrium vapor pressure over a curved water droplet, accounting for both curvature and solute effects.

Fractional reduction in the vapor pressure is equivalent to mole fraction.

Mole fraction of a pure water droplet is defined as number of moles of pure water in the solution by the total number of moles of the solution.

$$\mathscr{F} = \frac{n_{PW}}{n_{PW} + n_S} \tag{540}$$

Consider a solution of droplet of radius r that contains a mass m (in kg) dissolved in material of molecular mass M_S. For example, take solution of common salt in water:

$$NaCl \rightarrow Na^+ + Cl^-$$

Each molecule of the material dissolves in i ions in water.

$\therefore$ Effective number of moles will be:

$$n_{\text{eff}} = n_S = i\frac{m}{M_S} \tag{541}$$

If we assume ρ' is the density of solution, number of moles of pure water in the droplet is:

$$n_{\text{PW}} = \frac{\left(\frac{4}{3}\pi r^3 \rho' - M\right)}{M_w}$$

Mole fraction ($\mathscr{F}$) of water droplet, substitute Eq.(541) and Eq.(542) in Eq.(540):

$$\mathscr{F} = \frac{\left(\frac{4}{3}\pi r^3 \rho' - M\right)/M_w}{\left(\frac{4}{3}\pi r^3 \rho' - M\right)/M_w + im/M_S} \tag{542}$$

$$\mathscr{F} = \left[1 + \frac{imM_w}{M_S\left(\frac{4}{3}\pi r^3 \rho' - M\right)}\right]^{-1} \tag{543}$$

$$\frac{e}{e_s} = \exp\left(\frac{2\sigma}{nkTr}\right) \tag{544}$$

$$\frac{e}{e_s} \cdot \mathscr{F} = \exp\left(\frac{2\sigma}{nkTr}\right)\left[1 + \frac{imM_w}{M_S\left(\frac{4}{3}\pi r^3 \rho' - M\right)}\right]^{-1} \tag{545}$$

Course complete

Thank you!